§10.13 – Approximating Sums

We've seen that finding the exact sum of a convergent series $S = \sum_{n=1}^{\infty} a_n$ is often VERY hard. However, we can always approximate S using a partial sum:

$$\boxed{S \approx S_N = \sum_{n=1}^{N} a_n .}$$

Ex: We know that $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and hence can approximate S using S_{10} (for example):

$$S \approx S_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{10^2} \approx 1.5498$$

The error in this approximation is

$$\boxed{\text{Error} = S - S_N}$$

In this section, we'll explore two cases where we can estimate the size of this error.

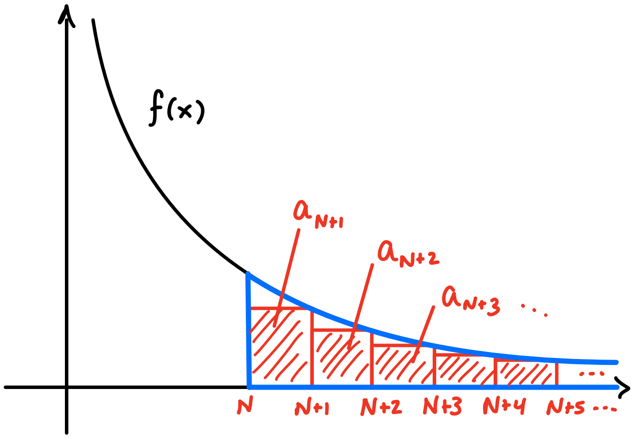
1. Series that Converge by the Integral Test

Suppose $S = \sum_{n=1}^{\infty} a_n$ converges by the integral test,

So $a_n = f(n)$, where $f(x)$ is continuous, positive, and decreasing for $x \in [1, \infty)$. We wish to estimate the error when approximating S by S_N.

$$\begin{aligned} \text{Error} &= S - S_N \\ &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \\ &= \sum_{n=N+1}^{\infty} a_n \\ &= a_{N+1} + a_{N+2} + a_{N+3} + \cdots \end{aligned}$$

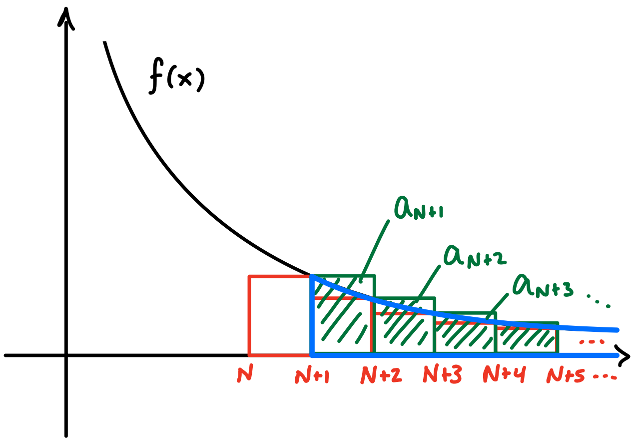
How big/small is this?



From the picture, we see that

$$\boxed{a_{N+1} + a_{N+2} + a_{N+3} + \cdots \leq \int_N^\infty f(x)\,dx}$$

Let's now view the picture slightly differently, using left endpoints for the rectangles instead of right:



From the picture, we see that

$$\boxed{a_{N+1} + a_{N+2} + a_{N+3} + \cdots \geq \int_{N+1}^\infty f(x)\,dx}$$

Combining these estimates, we have the following:

Integral Test Estimation Theorem (ITET)

If $\sum_{n=1}^{\infty} f(n)$ converges by the integral test, where f is continuous, positive, and decreasing, then the error in approximating the sum S using S_N satisfies

$$\boxed{\int_{N+1}^{\infty} f(x)\,dx \le S - S_N \le \int_N^{\infty} f(x)\,dx}$$

Ex: The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test (check as an exercise!) If we use S_{10} to approximate the sum, S, we get

$$S \approx S_{10} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \approx 1.5498.$$

(a) Estimate the error in this approximation

Solution: Let $f(x) = \frac{1}{x^2}$. We have

$$\int_{10+1}^{\infty} \frac{1}{x^2}\,dx \le S - S_{10} \le \int_{10}^{\infty} \frac{1}{x^2}\,dx$$

$$\Rightarrow \lim_{t\to\infty}\left[\frac{-1}{x}\right]_{11}^{t} \le S - S_{10} \le \lim_{t\to\infty}\left[\frac{-1}{x}\right]_{10}^{t}$$

$$\Rightarrow \lim_{t\to\infty}\left(\cancelto{0}{\frac{-1}{t}} + \frac{1}{11}\right) \le S - S_{10} \le \lim_{t\to\infty}\left(\cancelto{0}{\frac{-1}{t}} + \frac{1}{10}\right)$$

$$\Rightarrow \boxed{\underbrace{\frac{1}{11}}_{\approx 0.09} \le \underbrace{S - S_{10}}_{\text{Error}} \le \underbrace{\frac{1}{10}}_{=0.1}}$$

(b) How large or small could S possibly be?

Solution: From (a), we have

$$0.09 \le S - S_{10} \le 0.1$$

$$\Rightarrow \underset{+1.5498}{0.09} \le \underset{+1.5498}{S - 1.5498} \le \underset{+1.5498}{0.1}$$

$$\Rightarrow \boxed{1.6398 \le S \le 1.6498}$$

(The actual sum is $S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645$)

(c) How many terms N are needed in order to guarantee an error less than $\frac{1}{1000}$?

Solution: We have

$$\text{Error} = S - S_N \le \int_N^\infty \frac{1}{x^2}\,dx = \frac{1}{N}$$

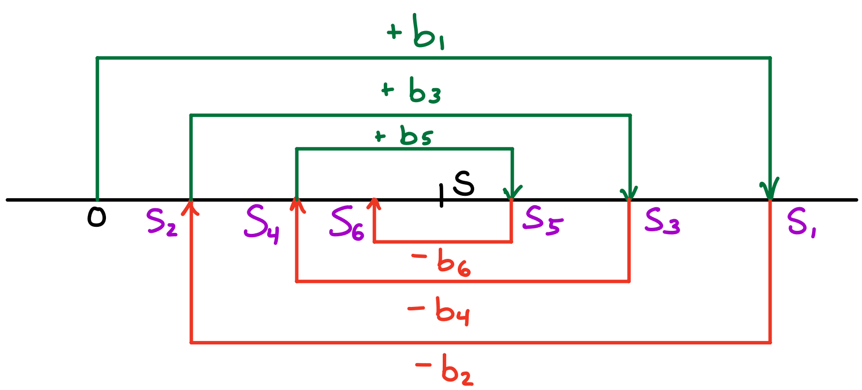
and so we want $\frac{1}{N} < \frac{1}{1000}$, or $N > 1000$.

Thus, $\boxed{N = 1001}$ will suffice.

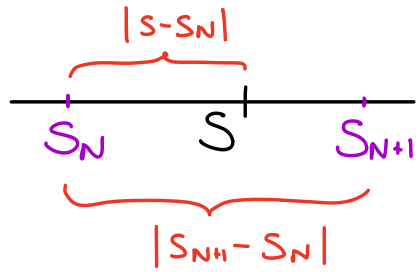
1. Series that Converge by the AST

Suppose that $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges by the AST,

so $b_n > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$.



Notice: S lies between any S_N and S_{N+1}.



Thus,

$$|\text{Error}| = |S - S_N| \leq |S_{N+1} - S_N|$$

$$= \underbrace{\left| \sum_{n=1}^{N+1} (-1)^{n+1} b_n - \sum_{n=}^{N} (-1)^{n+1} b_n \right|}_{\text{Everything except } (N+1)^{th} \text{ term cancels!}}$$

$$= \left| (-1)^{(N+1)+1} b_{N+1} \right|$$

$$= b_{N+1},$$

giving us the following result:

Alternating Series Estimation Theorem (ASET)

If $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges by the AST, then the error in approximating the sum S by S_N satisfies

$$|\text{Error}| = |S - S_N| \leq b_{N+1}$$

Ex: The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the AST.

(a) Estimate the size of the error when we use

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = 0.58\overline{3}$$

to approximate the sum, S.

Solution: Since $b_n = \frac{1}{n}$, the error satisfies

$$\underline{|\text{Error}| \leq b_{4+1} = b_5 = \frac{1}{5} \quad (\text{or } 0.2)}$$

[Note: Since $|\text{Error}| = |S - S_4| \leq 0.2$, we have

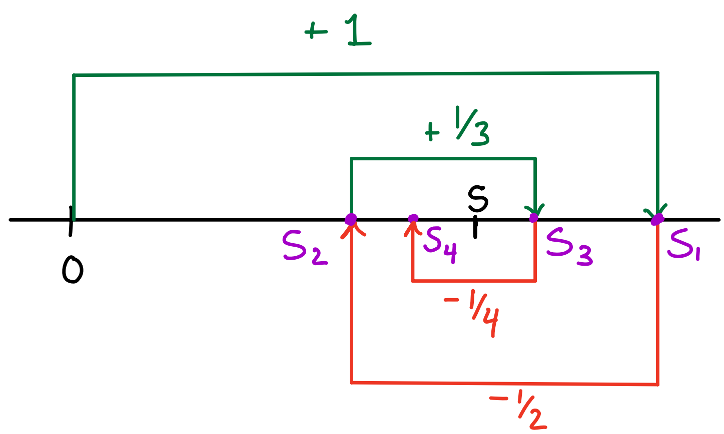
$$-0.2 \leq S - S_4 \leq 0.2$$

$$\Rightarrow \underset{+0.58\overline{3}}{-0.2} \leq \underset{+0.58\overline{3}}{S - 0.58\overline{3}} \leq \underset{+0.58\overline{3}}{0.2}$$

$$\Rightarrow \underline{0.38\overline{3} \leq S \leq 0.78\overline{3}}$$]

(b) Is S_4 an overestimate or underestimate of the sum, S?

Solution:



Since the final term in $S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ is negative,

the partial sum S_4 will underestimate S.

(So we can actually say $0.58\overline{3} \le S \le 0.78\overline{3}$)

Ex: The series $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges by the AST.

(a) How many terms N will guarantee that the partial sum S_N approximates S with $|\text{Error}| \le \frac{1}{10000}$?

Solution: With $b_n = \frac{1}{n!}$, the error satisfies

$$|\text{Error}| \le b_{N+1} = \frac{1}{(N+1)!}$$

Thus, we want

$$\frac{1}{(N+1)!} \leq \frac{1}{10000} \iff (N+1)! \geq 10000$$

It will be easiest to check small values of N:

N	3	4	5	6	7
$(N+1)!$	24	120	720	5040	40320

≥ 10000

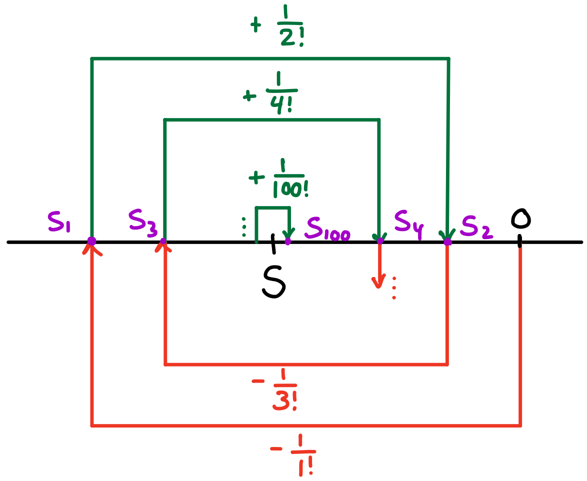
Thus we will need $N = 7$ terms!

(b) Is S_{100} an overestimate or underestimate of the overall sum, S?

Solution: Note that the final term in S_{100} is

$$\frac{(-1)^{100}}{100!} = \frac{1}{100!},$$

which is positive. This means S_{100} will overestimate S.



Additional Exercises:

Ex: Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)^2}$ converges by the integral test, then find an upper bound on the error when using S_{100} to approximate the sum, S.

Solution: Let $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$. We note that f is continuous, positive, and decreasing on $[1, \infty)$, hence the integral test applies. We have

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx$$

$u = \sqrt{x} + 1$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$= \lim_{t\to\infty} \int_2^{\sqrt{t}+1} \frac{2}{u^2}\,du$$

$$= \lim_{t\to\infty} \left[\frac{-2}{u} \right]_2^{\sqrt{t}+1}$$

$$= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t}+1} + \frac{2}{2} \right) = 1$$

Since $\int_1^\infty f(x)\,dx$ converges, so too does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(\sqrt{n}+1)^2}$.

For the error, note that

$$S - S_{100} \le \int_{100}^\infty \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx$$

$u = \sqrt{x} + 1$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$= \lim_{t\to\infty} \int_{\sqrt{100}+1}^{\sqrt{t}+1} \frac{2}{u^2}\,du$$

$$= \lim_{t\to\infty} \left[\frac{-2}{u} \right]_{11}^{\sqrt{t}+1}$$

$$= \lim_{t \to \infty} \left(\frac{-2}{\sqrt{t}+1} + \frac{2}{11} \right)$$

$$= \boxed{\frac{2}{11} = 0.\overline{18}}$$