④ Alternating Series

A series $\sum a_n$ is said to be alternating if the terms a_n alternate between positive and negative values.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ is alternating (it is known as the alternating harmonic series.)

Ex: $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$ is not considered to be alternating (sign must change from each term to the next!)

Below is a very simple test that can be used to show that certain alternating series converge.

The Alternating Series Test (AST)

Consider the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

where $b_n > 0$ for all n. If

(i) $\{b_n\}$ is a decreasing sequence, and

(ii) $\lim_{n\to\infty} b_n = 0$,

then $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges.

Let's apply the AST to the alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Here we have $b_n = \frac{1}{n}$. Note that

(i) $\{\frac{1}{n}\}$ is a decreasing sequence, and

(ii) $\lim_{n\to\infty} \frac{1}{n} = 0$, hence, by the AST, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges!

Remarks about the AST:

1. Will also work with series of the form $\sum_{n=1}^{\infty} (-1)^n b_n$, $\sum_{n=0}^{\infty} (-1)^{n-1} b_n$, etc. Just make sure the series is alternating!

2. If only (i) fails (i.e., $\{b_n\}$ is non-decreasing), the AST provides no information.

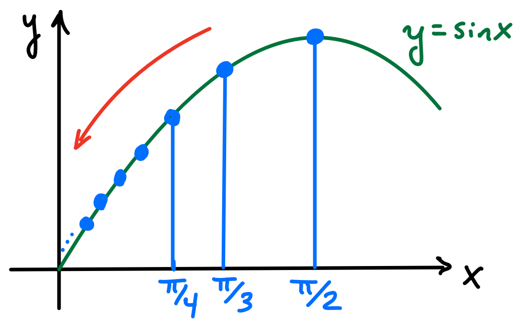
3. However, if (ii) fails (i.e., $\lim_{n\to\infty} b_n \neq 0$), then $\lim_{n\to\infty} (-1)^{n+1} b_n$ DNE, hence $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ diverges by the divergence test.

More Examples!

(a) $\sum_{n=2}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right) - \cdots$

Try the AST with $b_n = \sin(\pi/n)$. Note that

(i) $\{b_n\}$ is decreasing, as seen in the graph on the right.



[Alternatively, $f(x) = \sin(\pi/x)$ is decreasing since $f'(x) = \frac{-\pi}{x^2}\cos(\pi/x) < 0$.]

(ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin 0 = 0.$

Hence $\sum\limits_{n=2}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$ converges by the AST.

(b) $\sum\limits_{n=1}^{\infty} (-1)^n e^{1/n} = -e^1 + e^{1/2} - e^{1/3} + e^{1/4} - e^{1/5} + \cdots$

Try AST with $b_n = e^{1/n}$. Note that

(i) $\{b_n\}$ is decreasing, since $f(x) = e^{1/x}$ has derivative

$f'(x) = \frac{-1}{x^2} e^{\frac{1}{x}} < 0$ everywhere.

(ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} e^{1/n} = e^0 = 1$

uh oh... can't use AST! However, we can use the divergence test!

Since $\lim_{n\to\infty} (-1)^n e^{1/n}$ DNE (it jumps between ≈ 1 and ≈ -1),

$\sum_{n=1}^{\infty} (-1)^n e^{1/n}$ diverges by the divergence test.