§1.6 - Compositions and Inverses

Given two functions $f(x)$ and $g(x)$, the composition of f with g, denoted $f \circ g$, is the function

$$\boxed{(f \circ g)(x) = f(g(x))}$$

Similarly,

$$\boxed{(g \circ f)(x) = g(f(x))}$$

Ex: Let $f(x) = x^2 + 1$ and $g(x) = x + 7$.

$$(f \circ g)(1) = f(g(1)) = f(1+7) = f(8) = 8^2 + 1 = \boxed{65}$$

$$(g \circ f)(1) = g(f(1)) = g(1^2 + 1) = g(2) = 2 + 7 = \boxed{9}$$

$$(f \circ g)(x) = f(g(x)) = f(x+7) = \boxed{(x+7)^2 + 1}$$

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = (x^2+1) + 7 = \boxed{x^2 + 8}$$

Note: We can also talk about $f \circ f$, $g \circ g$, $g \circ f \circ g$, etc.

The domain of $f(g(x))$ is found as before (i.e., by removing any "problem points"), but now we must also account for the domain of $g(x)$, the inner function, by removing <u>its</u> "problem points".

<u>Ex:</u> Let $f(x) = \frac{x}{1+x}$, $g(x) = \frac{1}{x}$. Find $f(g(x))$ and its domain.

<u>Solution:</u> $f(g(x)) = f\left(\frac{1}{x}\right) = \dfrac{\frac{1}{x}}{1+\frac{1}{x}} = \dfrac{\frac{1}{x}}{\frac{x}{x}+\frac{1}{x}}$

Must exclude $x=0$ since it's not in the domain of g!

$$= \dfrac{\frac{1}{x}}{\frac{x+1}{x}}$$

Must exclude $x=-1$, else we divide by 0

$$= \frac{1}{\cancel{x}} \cdot \frac{\cancel{x}}{x+1}$$

$$= \boxed{\frac{1}{x+1}}$$

$$\boxed{\therefore \text{ Domain } = \{x \in \mathbb{R} : x \neq 0 \text{ and } x \neq -1\}}$$

Ex: If $f(x) = \sqrt{x} + 1$ and $g(x) = (x-1)^2 + 1$, find $g \circ f$ as well as its domain and range.

Solution: $g(f(x)) = g(\sqrt{x}+1) = ((\sqrt{x}+1)-1)^2 + 1$

$= (\sqrt{x})^2 + 1 = \boxed{x+1}$

Domain: All $x \in \mathbb{R}$, but we also need $x \geq 0$ for the domain of f, hence $\boxed{\text{domain} = [0, \infty).}$

Range: Since $y = g(f(x)) = x+1$ with $x \in [0, \infty)$,

$\boxed{\text{range} = [1, \infty).}$

Inverse Functions

Given an input x, a function $y = f(x)$ tells us the corresponding y-value. But what if we start with y and want the corresponding x?

Ex: $y = f(x) = \frac{x+3}{x}$. Find x such that $y=7$

Solution: $y = 7 = \frac{x+3}{x} \implies 7x = x+3$

$\implies 6x = 3$

$\implies x = 3/6 = \boxed{\frac{1}{2}}$

In fact, we can do this with any y:

$$y = \frac{x+3}{x} \implies xy = x+3$$

$$\implies xy - x = 3$$

$$\implies x(y-1) = 3 \implies \boxed{x = \frac{3}{y-1}}$$

This new function is called the inverse of f and is written $x = f^{-1}(y)$. It "undoes" the function f!

e.g. With $y = f(x) = \frac{x+3}{x}$, if $y=7$, then

$$x = f^{-1}(y) = \frac{3}{y-1} = \frac{3}{7-1} = \boxed{\frac{1}{2}}$$

(same as before!)

Note: We often like seeing x as the input variable and y as the output variable, even for inverses.

So, we usually swap x and y when calculating f^{-1}.

e.g. In our previous example, we have

$$f(x) = \frac{x+3}{x} \implies f^{-1}(x) = \frac{3}{x-1}$$

$f^{-1}(y) = \frac{3}{y-1}$

variable swap!

Ex: Find $f^{-1}(x)$ given $f(x) = \frac{2x}{3x-1}$.

Solution:

$$y = \frac{2x}{3x-1} \implies y(3x-1) = 2x$$

$$\implies 3xy - y = 2x$$

$$\implies x(3y-2) = y$$

$$\implies x = \frac{y}{3y-2}$$

Swapping the variables: $\boxed{y = f^{-1}(x) = \frac{x}{3x-2}}$

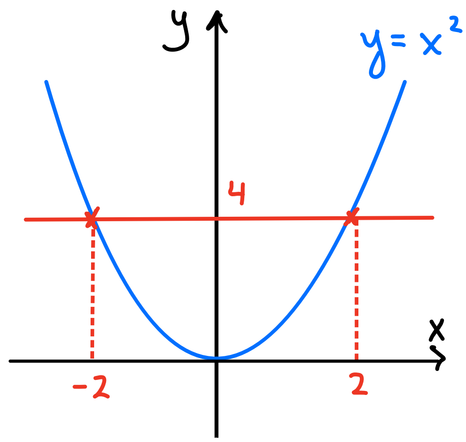
Since f^{-1} "undoes" the function f (and vice-versa):

$$\boxed{f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.}$$

Q: Do all functions have inverses? No!

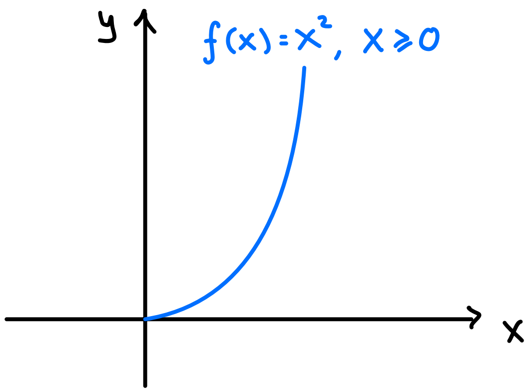
Ex: Consider $y = f(x) = x^2$.

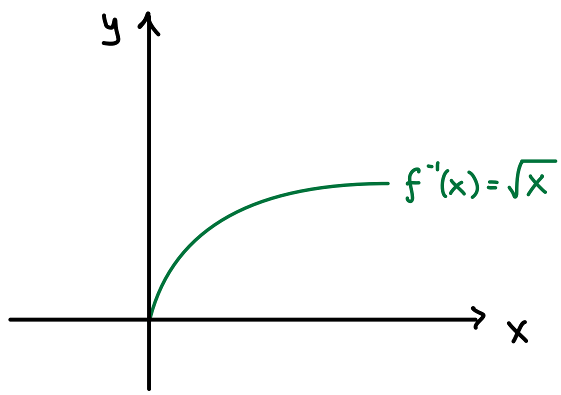
There is no way to "undo" f since $y=4$, for example, could have come from multiple x values: $x=2$ or $x=-2$.



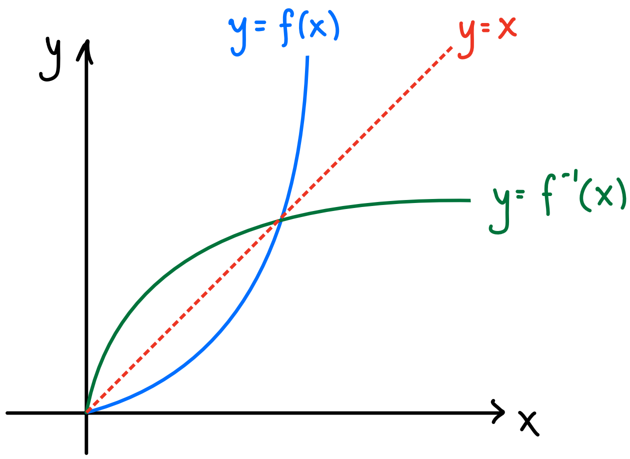
For a function $y = f(x)$ to have an inverse, it must pass the horizontal line test: every horizontal line intersects the graph of f at most once.

To define an inverse for $f(x) = x^2$, we would need to restrict its domain!





Graphically, $f^{-1}(x)$ can be obtained by reflecting the graph of $f(x)$ over the line $y = x$.



Useful facts:

- Domain of f = Range of f^{-1}
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Ex: What is the range of $f(x) = \dfrac{2x}{3x-1}$?

Solution: We showed earlier that $f^{-1}(x) = \dfrac{x}{3x-2}$

The domain of f^{-1} is $\{x \in \mathbb{R} : x \neq 2/3\}$ and

hence the range of f is $\boxed{\{y \in \mathbb{R} : y \neq 2/3\}}$.