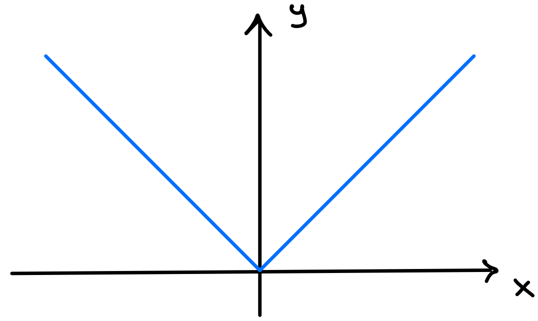
The Absolute Value Function

This is our first example of a piecewise-defined function!

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Ex: Write the following in piecewise form and sketch the graph.

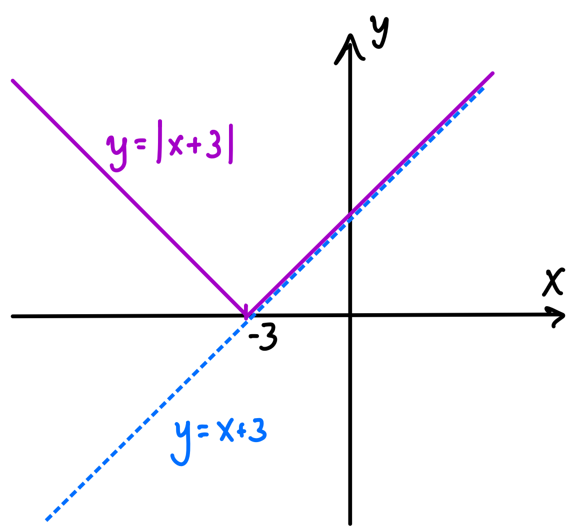
(a) $y = |x+3|$ (b) $y = |x^2 - 4|$

Solution:

(a)

$$|x+3| = \begin{cases} x+3 & \text{if } x+3 \geq 0 \\ -(x+3) & \text{if } x+3 < 0 \end{cases}$$

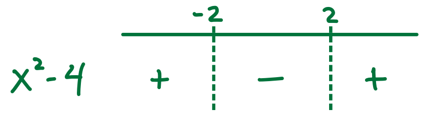
$$= \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \end{cases}$$



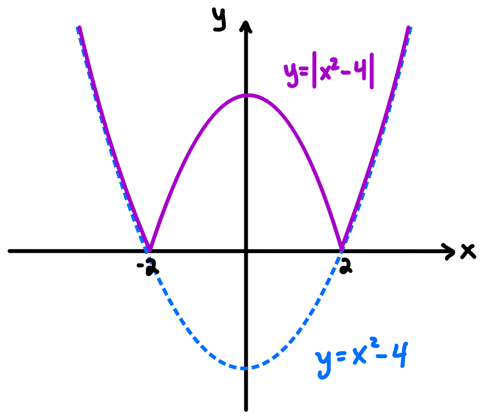
(b)

$$|x^2-4| = \begin{cases} x^2-4 & \text{if } x^2-4 \geq 0 \\ -(x^2-4) & \text{if } x^2-4 < 0 \end{cases}$$

$x^2-4 = (x-2)(x+2) = 0$ when $x=2$ or $x=-2$.



$$= \begin{cases} x^2-4 & \text{if } x \leq -2 \text{ or } x \geq 2 \\ 4-x^2 & \text{if } -2 < x < 2 \end{cases}$$



Properties of Absolute Value

$$|xy| = |x| \cdot |y| , \quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|} , \quad |x| = |-x|$$

which means $f(x) = |x|$ is even!

Note that $|x \pm y| \neq |x| \pm |y|$!

e.g.

$|-1+5| = |4| = 4$

$|-1|+|5| = 1+5 = 6$

Not equal

Instead, we have what's called the triangle inequality:

$$|x \pm y| \leq |x| + |y|$$

Absolute values may also show up when simplifying squares and square roots:

$$(\sqrt{x})^2 = x \quad \text{while} \quad \sqrt{x^2} = |x| \; !$$

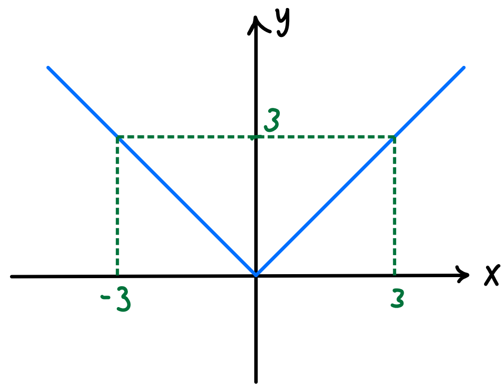
(For example, when $x = -1$, $\sqrt{(-1)^2} = \sqrt{1} = 1 = |-1|$)

Whether you realize it or not, we often use this fact when solving equations.

e.g., $x^2 = 9 \Rightarrow \sqrt{x^2} = \sqrt{9}$

$\Rightarrow |x| = 3$

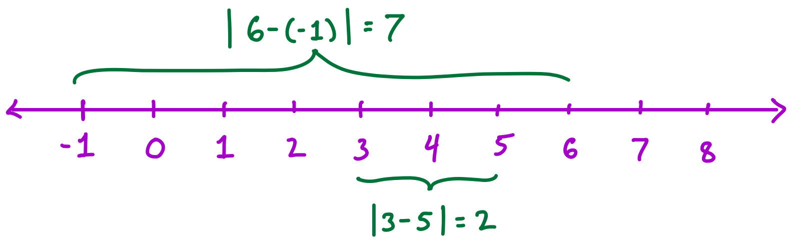
$\Rightarrow x = \pm 3.$



Geometrically, $|x - a|$ represents the distance between x and a on the real number line.

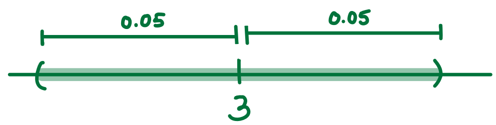
e.g. $|3-5| = 2$, the distance between 3 and 5.

$|6-(-1)| = |6+1| = 7$, the distance between 6 and -1.



Ex: Find all x such that $|x-3| < 0.05$.

Solution: If $|x-3| < 0.05$, then the distance between x and 3 is less than 0.05.



So $3-0.05 < x < 3+0.05$, or $2.95 < x < 3.05$

Thus, $x \in (2.95, 3.05)$.

In general,

$$|x-a| < \delta \iff x \in (a-\delta, a+\delta).$$

"if and only if"

This is useful when describing measurement errors or approximation errors (which we'll see a LOT in MATH 118.)

Specifically, if we write "$x \approx a$" (e.g., $\pi \approx 3$), the <u>absolute error</u> in this approximation is $|x-a|$, the distance between x and a.

To ensure our approximation errors are small, we'll often need to consider inequalities of the form

$$\boxed{|x-a| < \delta.}$$

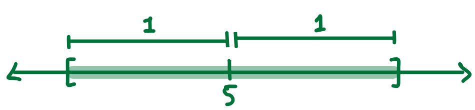
abs. error ($|x-a|$) — some small error tolerance (δ)

Ex: Solve $|2x-5| \leq 1$

Solution: $\underbrace{|2x-5| \leq 1}_{\text{"Distance between } 2x \text{ and } 5 \text{ is } \leq 1\text{"}}$ $\iff$ $5-1 \leq 2x \leq 5+1$

$\iff$ $4 \leq 2x \leq 6$ $(\div 2)$

$\iff$ $2 \leq x \leq 3$

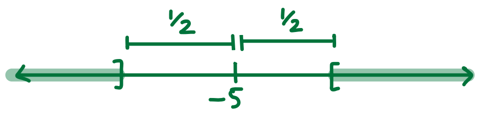


Therefore, $\boxed{x \in [2,3]}$

Ex: Solve $|x+5| \geq \frac{1}{2}$

Solution: $|x+5| \geq 1/2 \iff \underbrace{|x-(-5)| \geq 1/2}$

"Dist. from x to −5 is ≥ 1/2"



$\iff x \leq -5-\frac{1}{2}$ or $x \geq -5+\frac{1}{2}$

$\iff x \leq -\frac{11}{2}$ or $x \geq \frac{-9}{2}$

Thus, $\boxed{x \in (-\infty, -11/2] \cup [-9/2, \infty)}$

Ex: Solve $\left|\frac{x}{x-6}\right| > 1$

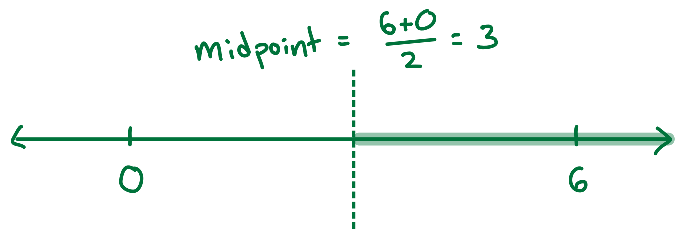
Solution: $\left|\frac{x}{x-6}\right| > 1 \iff \frac{|x|}{|x-6|} > 1$

$\cdot |x-6|$

$\iff \underbrace{|x|} > |x-6|$

$= |x-0|$, the distance from x to 0.

So, we're looking for all x whose distance to 0 is > its distance to 6. (i.e., closer to 6 than to 0.)



Also, $x \neq 6$ since the original inequality would be undefined.

Thus, $\boxed{x \in (3,6) \cup (6,\infty)}$