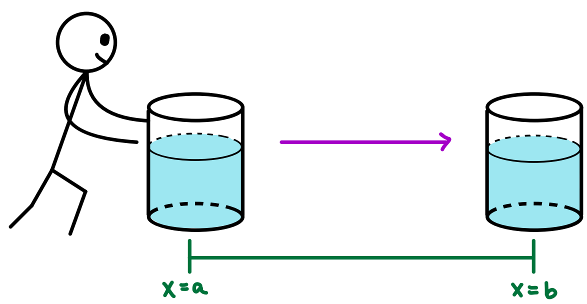
§ 7.4 - Work

Suppose a constant force F is applied to an object, moving it from $x=a$ to $x=b$.

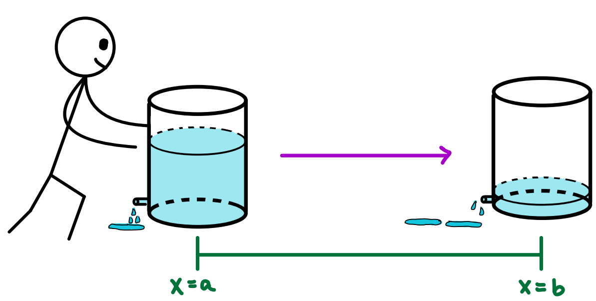


In this case, the work done on the object is

$$\boxed{\text{Work} = \text{Force} \cdot \text{Distance} = F \cdot (b-a)}$$

What if the force is not constant, and instead changes throughout the process (so force $= F(x)$)?

For example, if the container above were leaking...



... then it would get lighter as x increases, meaning less and less force would be needed to move it!

If force $= F(x)$, then the work done in moving the object a small distance $\Delta x \approx dx$ is roughly $F(x)\Delta x$.

Adding these work amounts from $x=a$ to $x=b$, we get

$$\boxed{\text{Work} = \int_a^b F(x)\,dx}$$

Ex: A 10m chain of mass 40kg hangs from the top of a building. How much work is done in lifting the chain to the top?

Solution: First, let's figure out how much mass is being moved at each point of the process.

The chain has density $\dfrac{40\,kg}{10\,m} = 4\ kg/m$ so once x metres have been lifted (and $10-x$ metres remain), the remaining chain has mass

$$4\,\frac{kg}{m}\cdot(10-x)\,m = 40-4x\ kg$$

Hence,

Due to gravity $= 9.81\ m/s^2$

$$\text{Force} = \text{Mass}\cdot\text{Acceleration}$$

$$\Rightarrow F(x) = (40-4x)\cdot 9.81\ N \quad (\text{Newtons})$$

Thus,

$$\text{Work} = \int_0^{10} F(x)\,dx = \int_0^{10}(40-4x)\cdot 9.81\,dx$$

$$= 9.81\left[40x-2x^2\right]_0^{10}$$

$$= 9.81\,(400-2\cdot 100)$$

$$= \boxed{1962\ J} \quad (\text{joules})$$

<u>Ex:</u> Rapunzel's hair has a constant density of 3 kg/m and hangs to the bottom of her 100 m tower. Set up the integrals that give each of the following.

(a) The work done in lifting the hair to the top of the tower.

<u>Solution:</u> Once x metres have been lifted (and 100 − x metres remain), we have

$$m_{hair} = 3\,\frac{kg}{m}\cdot(100-x)\,m = 300-3x \text{ kg}.$$

Hence,

$$\text{Force} = \text{Mass}\cdot\text{Acceleration}$$

$$\Rightarrow F(x) = (300-3x)\cdot 9.81 \quad N$$

Thus,

$$\boxed{\text{Work} = \int_0^{100} F(x)\,dx = \int_0^{100} (300-3x)\cdot 9.81\,dx}$$

(b) The work done in lifting the hair the first half of the way up.

Solution: We need only modify the bounds from (a).

$$\boxed{\text{Work} = \int_0^{50} F(x)\,dx = \int_0^{50} (300-3x)\cdot 9.81\,dx}$$

(c) The work done in lifting the hair to the top, given that her 10 kg "Skip the Dishes" order is tied to the end.

Solution: Once x metres of hair have been lifted,

$$m_{hair} = 300 - 3x \text{ kg}, \qquad m_{food} = 10 \text{ kg (constant)}$$

Thus,

$$\text{Force} = \text{Mass} \cdot \text{Acceleration}$$

$$\Rightarrow \quad F(x) = \Big[\underbrace{(300-3x)}_{m_{hair}} + \underbrace{10}_{m_{food}}\Big]\cdot 9.81 \text{ N}$$

Hence, we get

$$\boxed{\text{Work} = \int_0^{100} F(x)\,dx = \int_0^{100} (310-3x)\cdot 9.81\,dx}$$

(d) Same as (c), but the "Skip the Dishes" order falls off after being lifted 10 m.

<u>Solution:</u> In the first 10 m, we have

$$\text{Work} = \int_0^{10} (\underbrace{310}_{m_{hair} + m_{food}}-3x)\cdot 9.81\,dx$$

In the last 90 m, we have

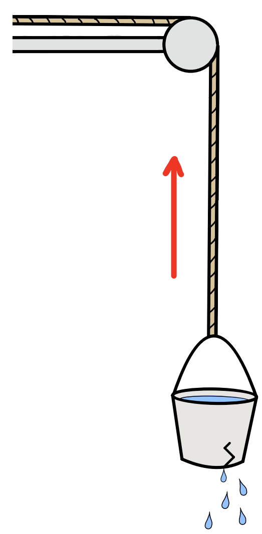
$$\text{Work} = \int_{10}^{100} (\underbrace{300}_{m_{hair}}-3x)\cdot 9.81\,dx$$

Hence, the total work is

$$\boxed{\int_0^{10} (310-3x)\cdot 9.81\,dx + \int_{10}^{100} (300-3x)\cdot 9.81\,dx}$$

Ex: A 50m rope of density 2kg/m hangs into a well. The rope is attatched to a 10kg bucket initially filled with 6kg of water. If water leaks from the bucket at a constant rate and only 1kg of water remains when the bucket reaches the top of the well, determine the work needed to lift the bucket to the top. Assume acceleration due to gravity is $10 m/s^2$.

Solution: Over the 50m of distance, $6-1 = 5$kg of water are lost, so the water is leaking at a rate of $\frac{5 kg}{50 m} = \frac{1}{10} \frac{kg}{m}$. Thus, once x metres of rope have been lifted and $(50-x)$ metres remain, we have



$$m_{rope} = 2\,\frac{kg}{m} \cdot (50-x)\,m = 100-2x\ kg,$$

$$m_{bucket} = 10\ kg \quad (\text{constant}),$$

$$m_{water} = 6\,kg - \frac{1}{10}\,\frac{kg}{m} \cdot x\,m = 6 - \frac{x}{10}\ kg$$

Thus, we obtain

$$\text{Force} = \text{Mass} \cdot \text{Acceleration}$$

$$\Rightarrow \quad F(x) = \left[(100-2x) + 10 + \left(6 - \frac{x}{10}\right)\right] \cdot 10$$

$$= 1160 - 21x \quad N$$

Therefore,

$$\text{Work} = \int_0^{50} F(x)\,dx$$

$$= \int_0^{50} (1160 - 21x)\,dx$$

$$= \left[1160x - \frac{21x^2}{2}\right]_0^{50} = \boxed{31\,750\ J}$$