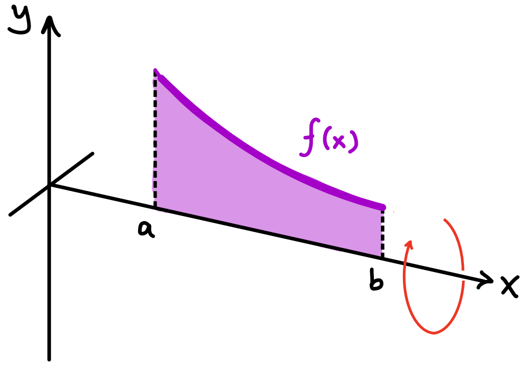
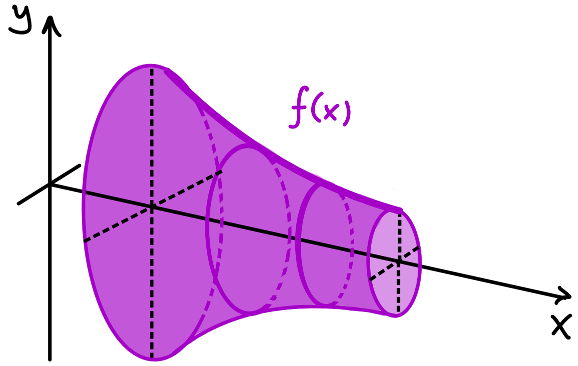
§7.2 – Volumes of Solids of Revolution

Integrals don't just tell us about areas, we can also use them for calculating volumes of 3D Solids!

The solids we'll consider in MATH 116 are called solids of revolution, and are obtained by revolving a 2D region about an axis:

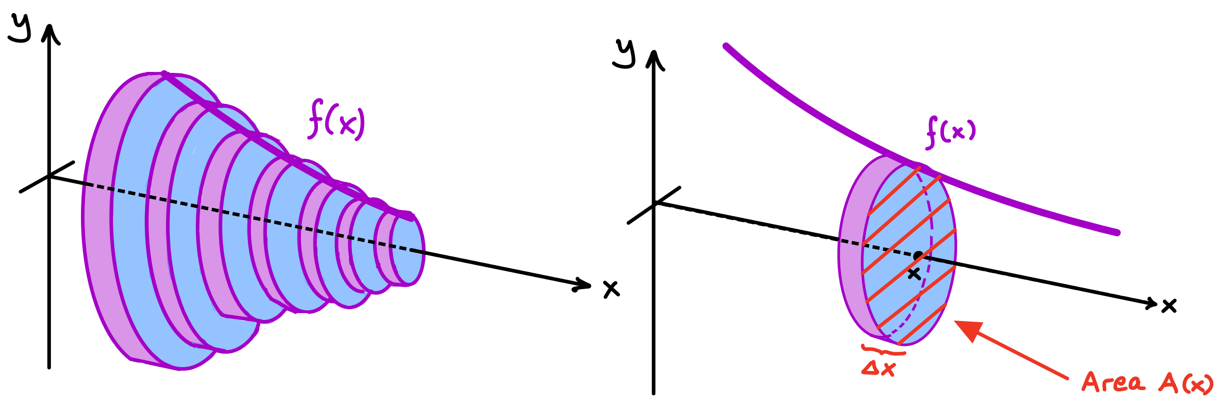




There are two ways to find the volume of such a solid.

1. The Disk / Washer Method

Start by slicing the solid into thin disks.



Each disk has width Δx. If $A(x)$ denotes the area of the disk at each point x, then the volume of a typical disk is $A(x)\Delta x$. Adding these volumes:

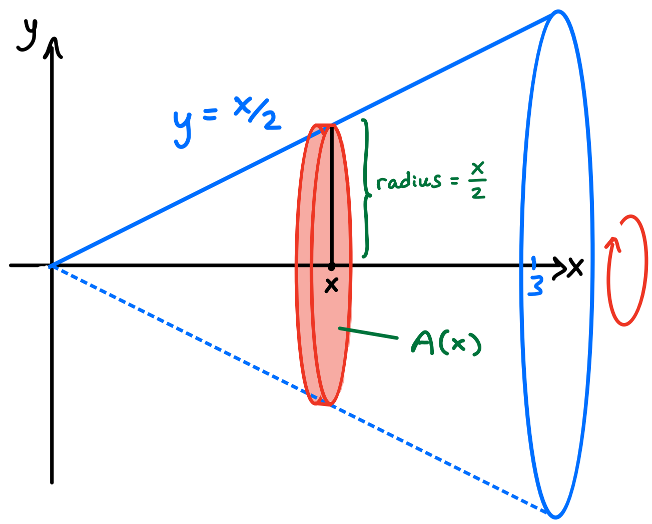
$$\boxed{\text{Volume of the Solid} = \int_a^b A(x)\,dx}$$

Ex: Consider the region between the x-axis and $y = \frac{x}{2}$ from $x = 0$ to $x = 3$. Find the volume of the solid obtained by rotating this region about the x-axis.

Solution: Start with a sketch showing the region and a typical disk. The area of a disk is

$A(x) = \pi \cdot \text{radius}^2 = \pi \cdot \left(\frac{x}{2}\right)^2,$

hence

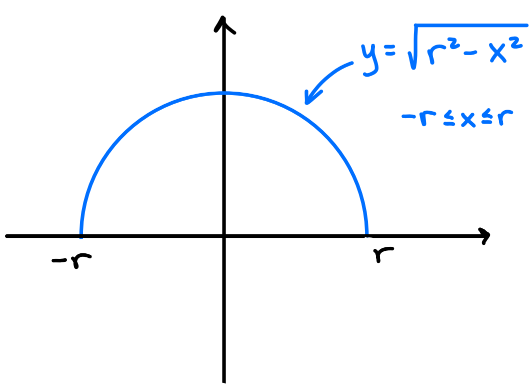


$$\text{Volume} = \int_0^3 A(x)\,dx = \int_0^3 \pi \left(\frac{x}{2}\right)^2 dx$$

$$= \frac{\pi}{4}\int_0^3 x^2\,dx = \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^3 = \boxed{\frac{9\pi}{4}}$$

Ex: Find the volume of a sphere of radius r.

Solution: A sphere can be obtained by rotating the top half of the circle $x^2+y^2=r^2$ about the x-axis!



$$\text{Volume} = \int_{-r}^{r} A(x)\,dx$$

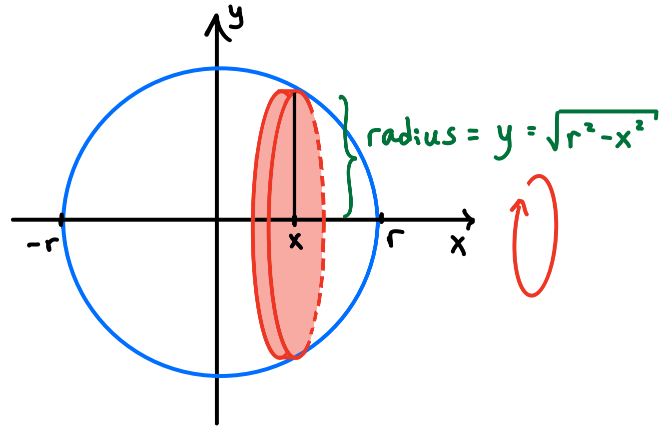
$$= \int_{-r}^{r} \pi \cdot \text{radius}^2 \,dx$$

$$= \int_{-r}^{r} \pi \left(\sqrt{r^2-x^2}\right)^2 dx$$

$$= \int_{-r}^{r} \pi\left(r^2-x^2\right) dx$$

$$= \left[\pi r^2 x - \frac{\pi x^3}{3}\right]_{-r}^{r}$$

$$= \left(\pi r^3 - \frac{\pi r^3}{3}\right) - \left(-\pi r^3 + \frac{\pi r^3}{3}\right) = \boxed{\frac{4\pi r^3}{3}}$$

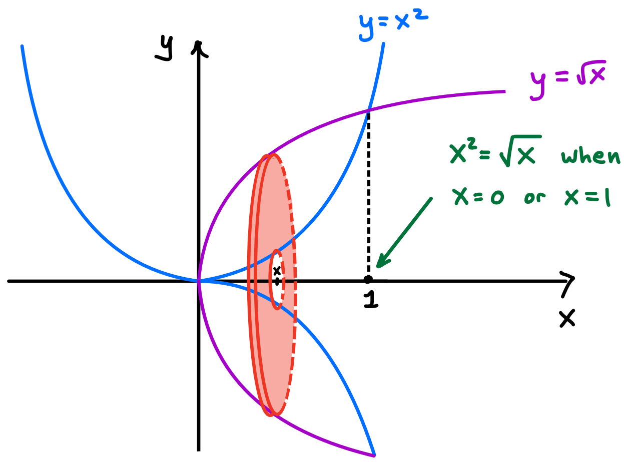


Ex: Set up but DO NOT EVALUATE the integral that gives the volume of the solid obtained by rotating each region about the given axis.

(a) Region: bounded between $y = x^2$ and $y = \sqrt{x}$

Axis: x-axis.

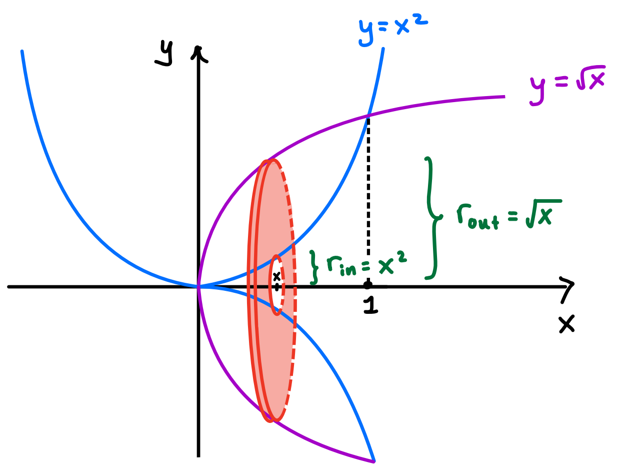
Solution: Start with a sketch!



This time our cross-section isn't a disk... it's a washer!

In this case,

$$\boxed{\text{Area} = A(x) = \pi \cdot (\text{outer radius})^2 - \pi(\text{inner radius})^2}$$



Outer radius $= \sqrt{x}$

Inner radius $= x^2$

Bounds: $0 \le x \le 1$

$$\therefore \text{Volume} = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(r_{out})^2 - \pi(r_{in})^2\right]dx$$

$$= \boxed{\int_0^1 \left[\pi(\sqrt{x})^2 - \pi(x^2)^2\right]dx}$$

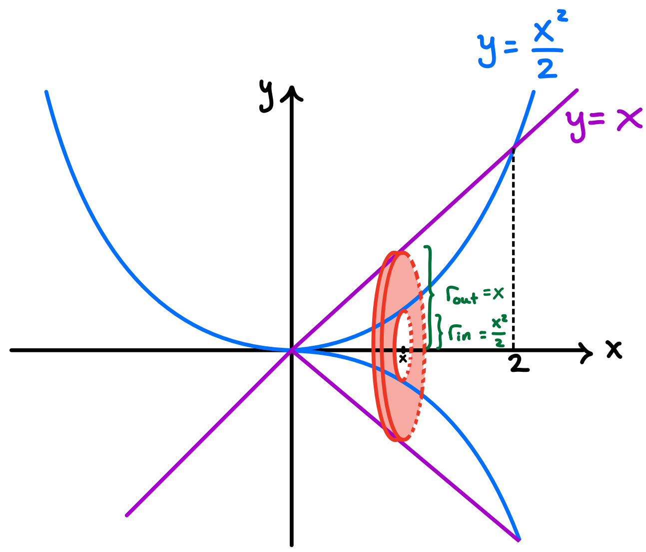
(b) Region: bounded between $y = x$ and $y = \frac{x^2}{2}$

Axis: x-axis.

Solution: Again, start with a sketch

Outer radius: x

Inner radius: $\frac{x^2}{2}$



Bounds?

$$\frac{x^2}{2} = x \Rightarrow x^2 = 2x$$

$$\Rightarrow x(x-2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 2.$$

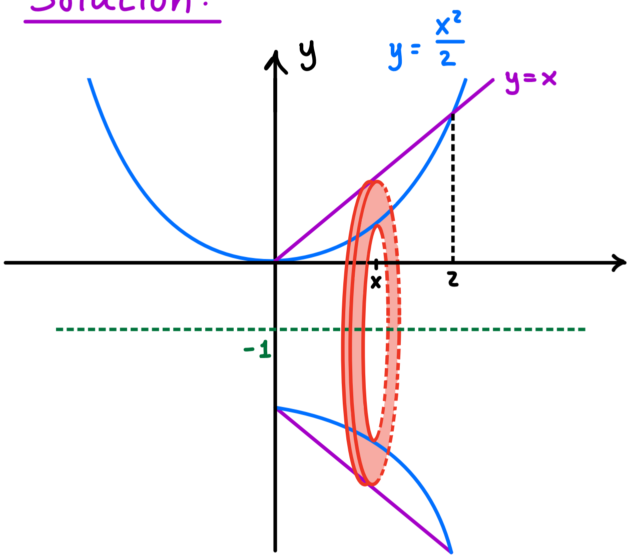
$$\therefore \text{Volume} = \int_0^2 A(x)\,dx = \int_0^2 \left[\pi (r_{out})^2 - \pi (r_{in})^2\right] dx$$

$$= \boxed{\int_0^2 \left[\pi (x)^2 - \pi \left(\frac{x^2}{2}\right)^2\right] dx}$$

(c) Region: bounded between $y = x$ and $y = \frac{x^2}{2}$

Axis: $y = -1$

Solution:



Outer radius $= 1+x$

Inner radius $= 1+\frac{x^2}{2}$

Bounds: $0 \le x \le 2$

(same as before)

$$\therefore \text{Volume} = \int_0^2 A(x)\,dx = \boxed{\int_0^2 \left[\pi(1+x)^2 - \pi\left(1+\frac{x^2}{2}\right)^2\right]dx}$$

(d) Region: bounded between $y=x$ and $y=\frac{x^2}{2}$

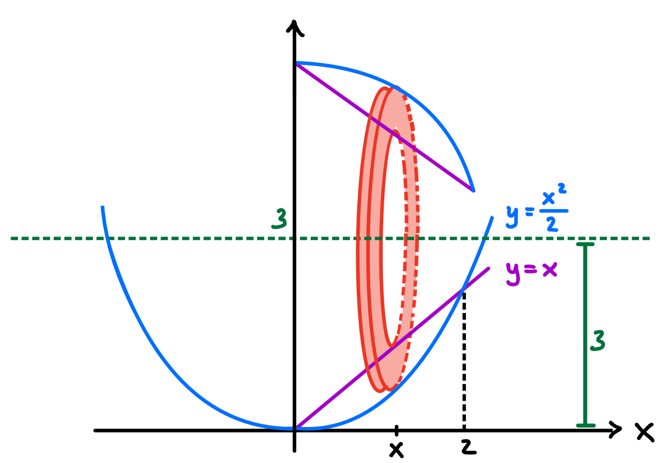
Axis: $y=3$

Solution:

Outer radius: $3-\frac{x^2}{2}$

Inner radius: $3-x$

Bounds: $0 \le x \le 2$.

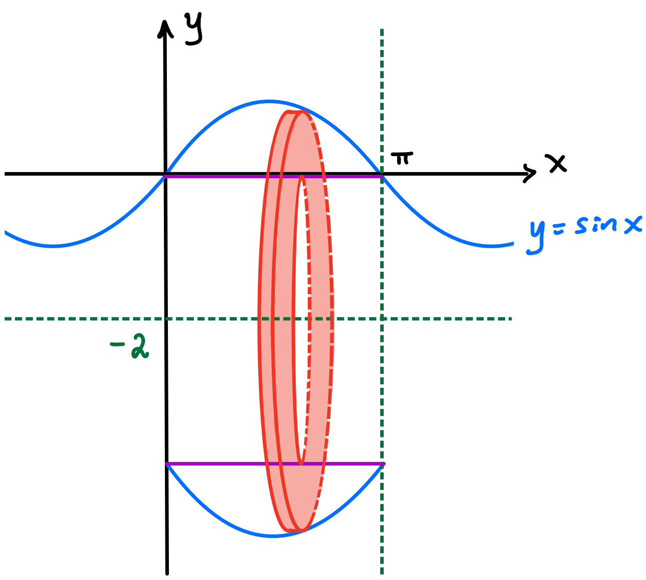


$\therefore$ Volume $= \int_0^2 A(x)\,dx = \boxed{\int_0^2 \left[\pi\left(3-\frac{x^2}{2}\right)^2 - \pi(3-x)^2\right] dx}$

(e) Region: bounded between $y = \sin x$ and $y = 0$ from $x = 0$ to $x = \pi$

Axis: $y = -2$

Solution:



Outer radius: $2 + \sin x$

Inner radius: 2

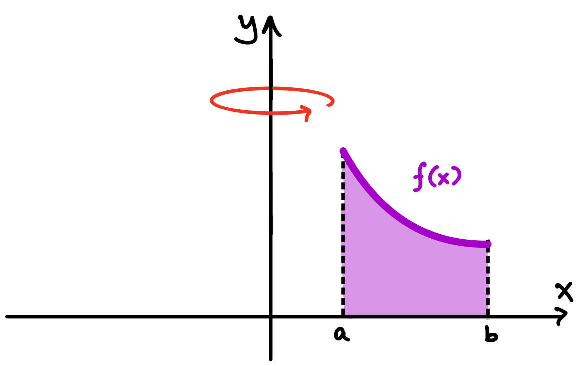
Bounds: $0 \le x \le \pi$

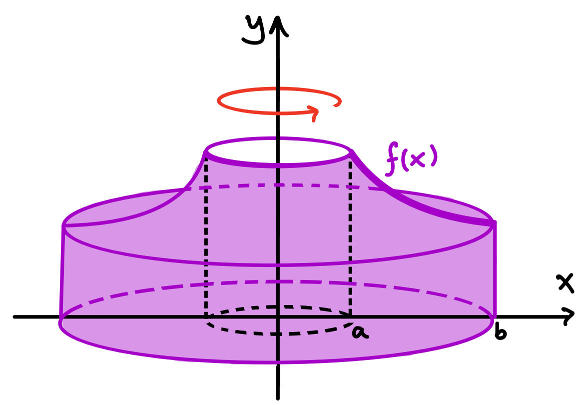
$\therefore$ Volume $= \int_0^{\pi} A(x)\,dx = \boxed{\int_0^{\pi} \left[\pi(2+\sin x)^2 - \pi(2)^2\right] dx}$

Our second method for calculating volumes will help when rotating about vertical axes.

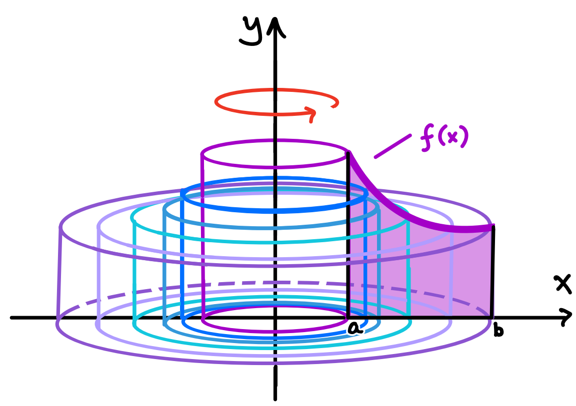
2. The Cylindrical Shell Method

Suppose we wish to compute the volume of the solid of revolution below.





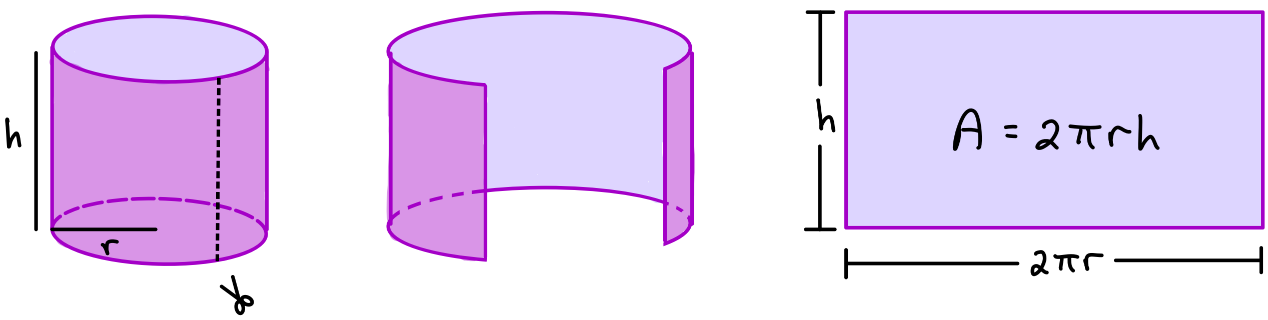
This time, we'll split up the solid into very thin cylindrical shells.



We again have

$$\boxed{\text{Volume} = \int_a^b A(x)\,dx}$$

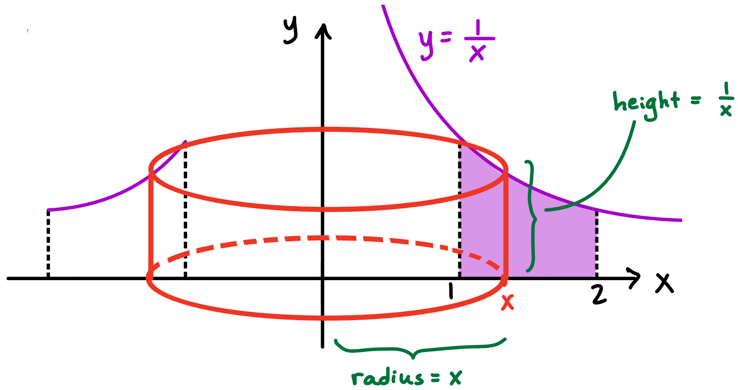
but now $A(x)$ is the surface area of a typical cylindrical shell.



$$\boxed{A(x) = 2\pi \cdot r \cdot h}$$

<u>Ex:</u> Let R denote the region between $y=0$ and $y=\frac{1}{x}$ from $x=1$ to $x=2$. Find the volume of the solid obtained by rotating R about the y-axis.

Solution:



$$\text{Volume} = \int_1^2 A(x)\,dx$$

$$= \int_1^2 2\pi r h\,dx$$

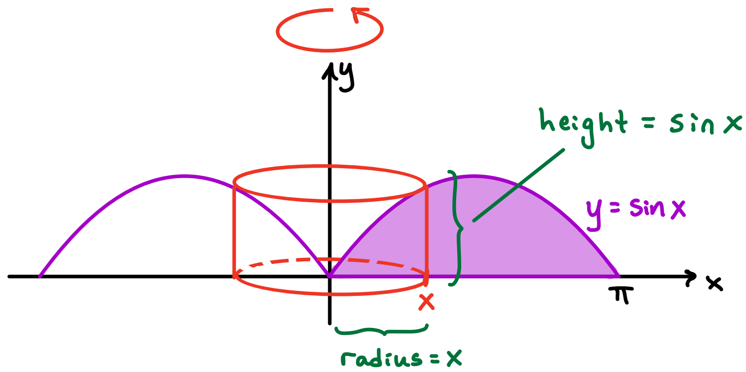
$$= \int_1^2 2\pi \cdot x \cdot \frac{1}{x}\,dx = \int_1^2 2\pi\,dx = \boxed{2\pi}$$

Ex: Set up but DO NOT EVALUATE the integral that gives the volume of the solid obtained by rotating each region about the given axis.

(a) Region: Between $y = \sin x$ and $y = 0$, $0 \le x \le \pi$

Axis: y-axis.

Solution:

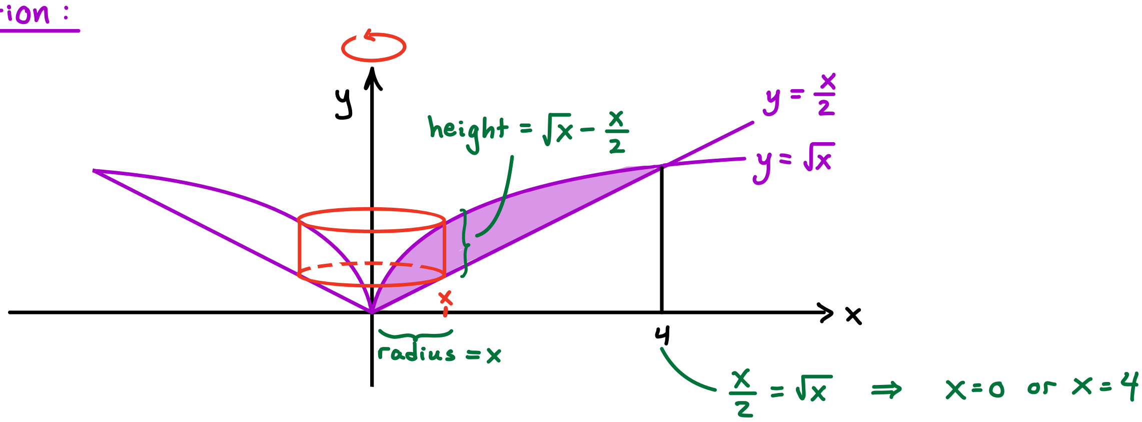


$$\text{Volume} = \int_0^{\pi} A(x)\,dx = \int_0^{\pi} 2\pi r h\,dx = \boxed{\int_0^{\pi} 2\pi x \cdot \sin x\,dx}$$

(b) Region: Bounded between $y = \sqrt{x}$ and $y = \frac{x}{2}$.

Axis: y-axis.

Solution:

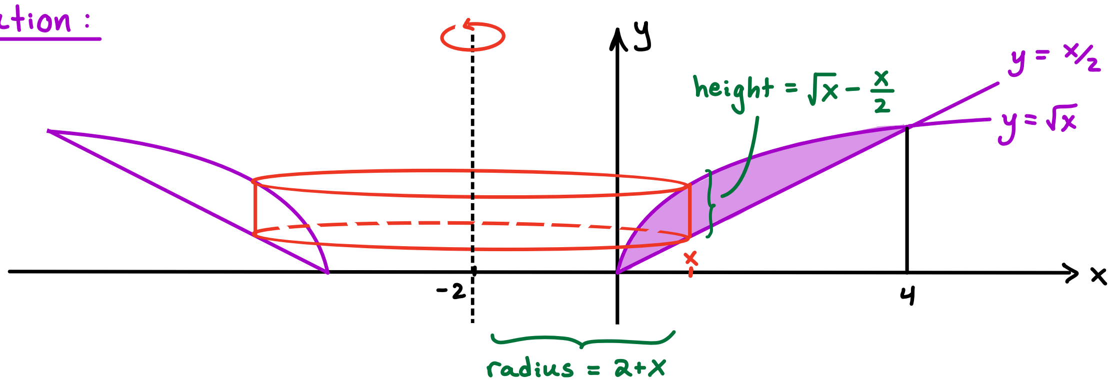


$$V = \int_0^4 A(x)\,dx = \int_0^4 2\pi r h\,dx = \boxed{\int_0^4 2\pi x\left(\sqrt{x} - \frac{x}{2}\right)dx}$$

(c) Region: Bounded between $y=\sqrt{x}$ and $y=\frac{x}{2}$.

Axis: $x=-2$

Solution:

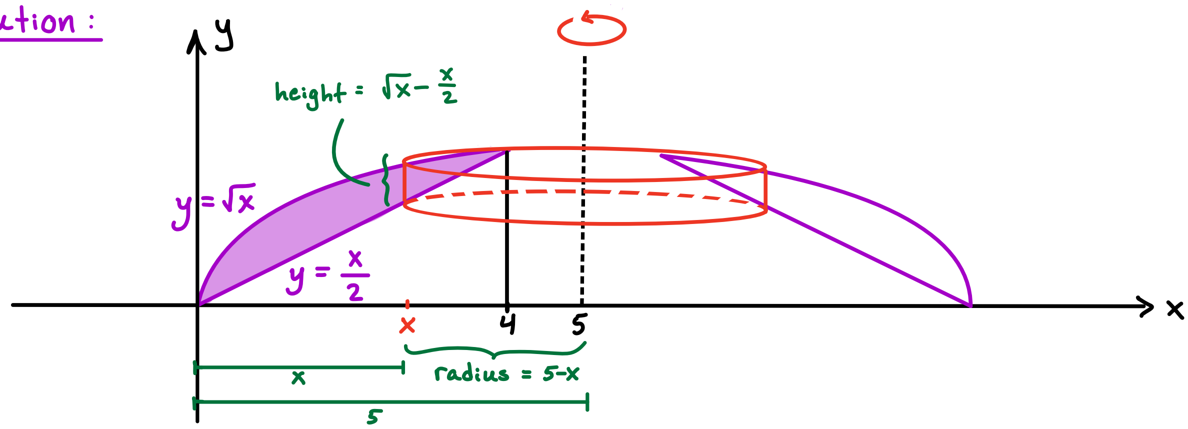


$$V = \int_0^4 A(x)\,dx = \int_0^4 2\pi r h\,dx = \boxed{\int_0^4 2\pi\,(2+x)\left(\sqrt{x}-\frac{x}{2}\right)dx}$$

(d) Region: Bounded between $y=\sqrt{x}$ and $y=\frac{x}{2}$.

Axis: $x=5$

Solution:



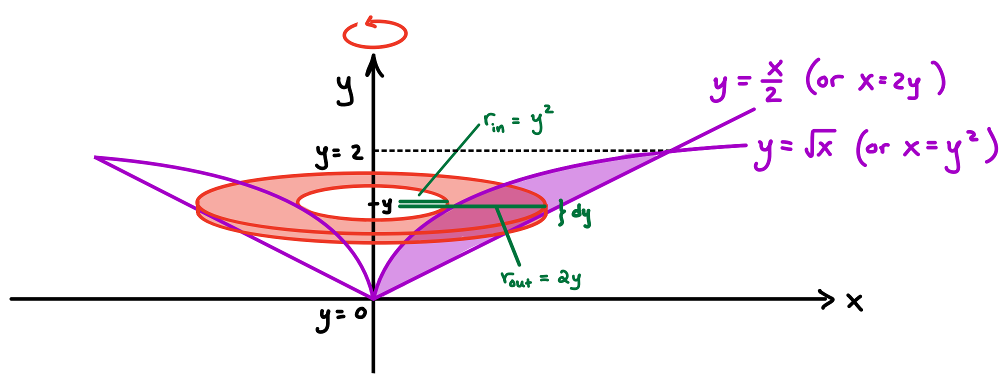
$$V = \int_0^4 A(x)\,dx = \int_0^4 2\pi r h\,dx = \boxed{\int_0^4 2\pi\,(5-x)\left(\sqrt{x} - \frac{x}{2}\right)dx}$$

As a final example, let's see how disks / washers when rotating about a vertical axis!

(e) Region: Bounded between $y=\sqrt{x}$ and $y=\frac{x}{2}$.

Axis: y-axis —— but use disks / washers!

Solution:



$$V = \int_{y=0}^{y=2} (\text{Area of washer})\,dy$$

$$= \int_0^2 \left(\pi (r_{out})^2 - \pi (r_{in})^2\right) dy = \boxed{\int_0^2 \left(\pi (2y)^2 - \pi (y^2)^2\right) dy}$$