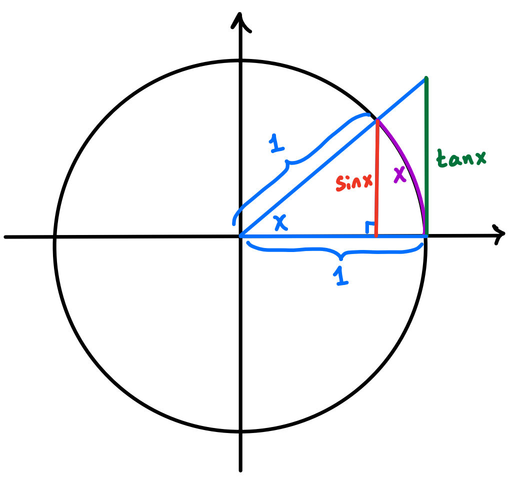
§3.9 - Derivatives of Trig Functions

To find the derivative of $f(x) = \sin(x)$, we'll need two very special limits!

See below why these are true!

$$\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0.}$$

Let's start with $\lim_{x\to 0} \frac{\sin x}{x}$!



From the diagram, we see that

$$\sin x \le x \le \tan x, \quad x \in (0, \tfrac{\pi}{2})$$

Allowing for negative x, we have

$$|\sin x| \le |x| \le |\tan x|, \quad x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

$$\Rightarrow \quad 1 \le \frac{x}{|\sin x|} \le \frac{|\tan x|}{|\sin x|} \qquad (\text{dividing by } |\sin x|)$$

$$\Rightarrow \quad 1 \ge \left|\frac{\sin x}{x}\right| \ge \Big|\underbrace{\frac{\sin x}{\tan x}}_{= \cos x}\Big| \qquad (\text{taking reciprocals})$$

[Can remove absolute values, as $\frac{\sin x}{x} > 0$ & $\cos x > 0$ as $x \to 0$.]

$$\implies \cos x \le \frac{\sin x}{x} \le 1$$

$$\implies \underbrace{\lim_{x\to 0} \cos x}_{=1} \le \lim_{x\to 0} \frac{\sin x}{x} \le \underbrace{\lim_{x\to 0} 1}_{=1}$$

By the squeeze theorem, $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

What about $\lim_{x\to 0} \frac{\cos x - 1}{x}$?

$$\lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\frac{\sin x}{\cos x + 1}}_{\to 0/2 = 0.}$$

$$= 1 \cdot 0 = \boxed{0.}$$

We may now calculate f' for $f(x) = \sin(x)$!

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\left[\cos(h) - 1\right] + \cos(x)\cdot\sin(h)}{h}$$

$$= \lim_{h \to 0} \sin(x)\cdot\left[\cancelto{0}{\frac{\cos(h)-1}{h}}\right] + \cos(x)\cancelto{1}{\frac{\sin(h)}{h}} = \boxed{\cos(x)}$$

Therefore,

$$\boxed{(\sin x)' = \cos x}$$

Similarly,

$$\boxed{(\cos x)' = -\sin x}$$

We can now find the derivatives of $\tan x$, $\sec x$, $\cot x$ and $\csc x$ using derivative rules!

Ex: Find $(\tan x)'$ and $(\sec x)'$.

Solution:

Quotient rule!

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \overbrace{(\sin x)'}^{\cos x} - \sin x \cdot \overbrace{(\cos x)'}^{-\sin x}}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\cos x \cdot \overbrace{1'}^{0} - 1 \cdot \overbrace{(\cos x)'}^{-\sin x}}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \boxed{\sec x \cdot \tan x}$$

Summary:

$(\sin x)' = \cos x$ $\qquad$ $(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$ $\qquad$ $(\sec x)' = \sec x \cdot \tan x$

$(\cot x)' = -\csc^2 x$ $\qquad$ $(\csc x)' = -\csc x \cdot \cot x$

Exercises!

Ex: Find the derivative of each function below.

(a) $y = \sec(1+3x)$

(b) $y = \sin^2(6x)$

Solution:

(a) $y' = \sec(1+3x)\tan(1+3x)\cdot(1+3x)'$ (chain rule)

$= \boxed{\sec(1+3x)\tan(1+3x)\cdot 3}$

(b) $y' = 2\sin(6x)\cdot[\sin(6x)]'$ (chain rule)

$= 2\sin(6x)\cdot\cos(6x)\cdot(6x)'$ (chain rule... again!)

$= \boxed{2\sin(6x)\cos(6x)\cdot 6}$