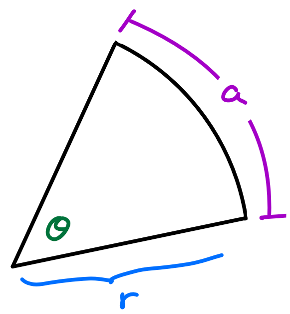
§1.7 - Trigonometry Review

In Calculus, angles are measured in radians.

A circle has 2π radians. A sector of a circle with radius r and central angle of θ radians has an arc length of

$$\boxed{a = \theta \cdot r}$$



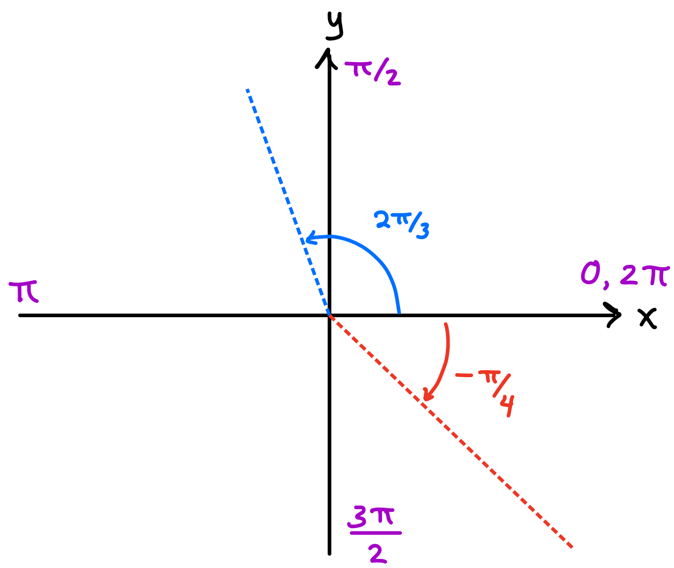
To convert between degrees and radians, we use the formulas

$$\boxed{\begin{aligned} \text{degrees} &= \left(\frac{180}{\pi}\right) \cdot \text{radians} \\ \text{radians} &= \left(\frac{\pi}{180}\right) \cdot \text{degrees} \end{aligned}}$$

Ex: $120°$ in radians is $120 \cdot \left(\frac{\pi}{180}\right) = \boxed{\frac{2\pi}{3}}$

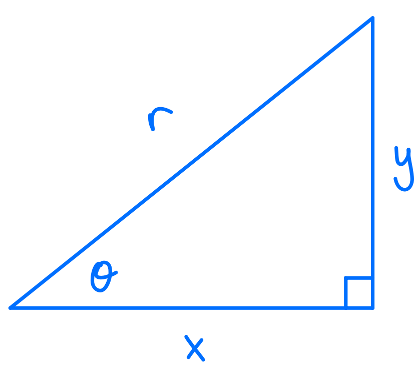
In the xy-plane, we usually measure angles counterclockwise from the positive x-axis.

Angles measured clockwise are considered negative.



Trigonometric Functions

Given a right triangle as below we define



$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

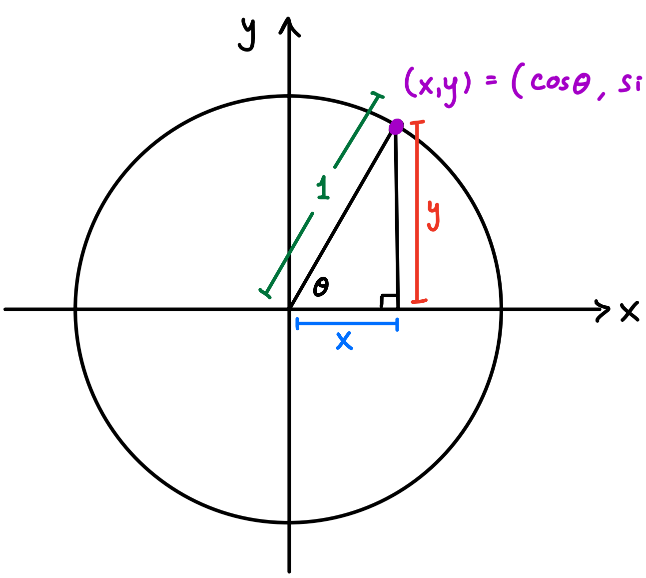
$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

We also define $\csc\theta = \frac{r}{y} = \frac{1}{\sin\theta}$

$$\sec\theta = \frac{r}{x} = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}.$$

From these definitions of $\sin\theta$ and $\cos\theta$, we see that any point on the unit circle below has coordinates $x = \frac{x}{1} = \cos\theta$ and $y = \frac{y}{1} = \sin\theta$.



By the Pythagorean theorem, $x^2 + y^2 = 1^2$, hence

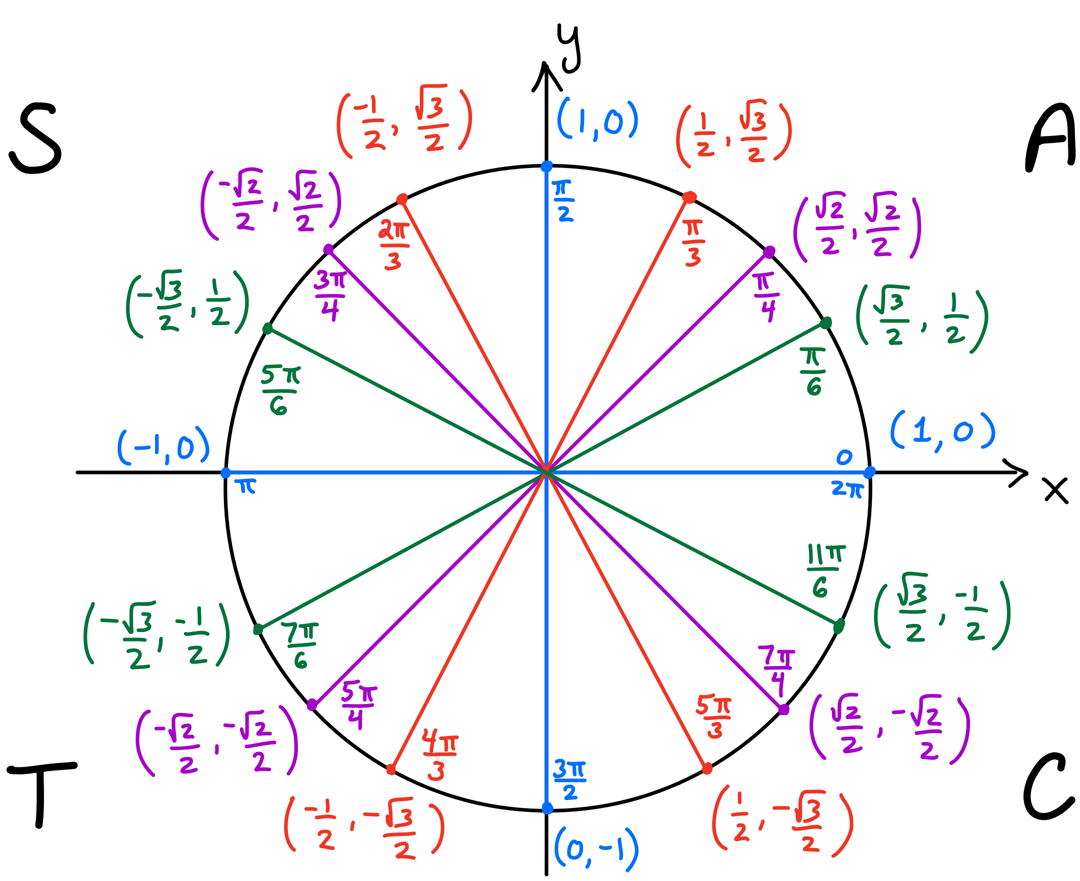
$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

From this we also get

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \implies \boxed{\tan^2\theta + 1 = \sec^2\theta}$$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \implies \boxed{1 + \cot^2\theta = \csc^2\theta}$$

You should memorize the value of $(\cos\theta, \sin\theta)$ for each "special angle" on the unit circle below.

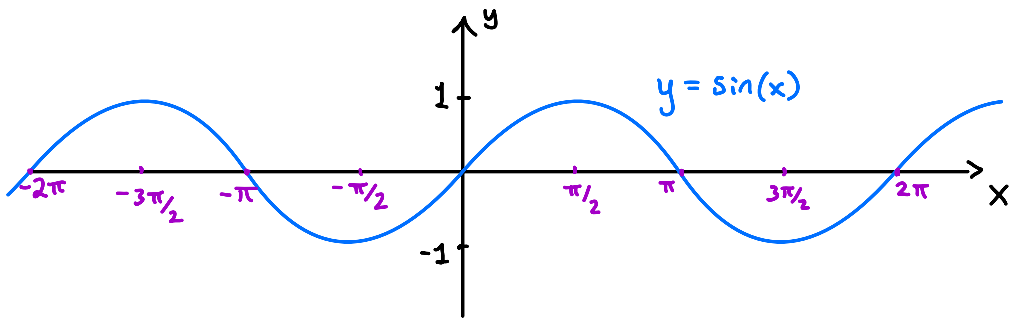


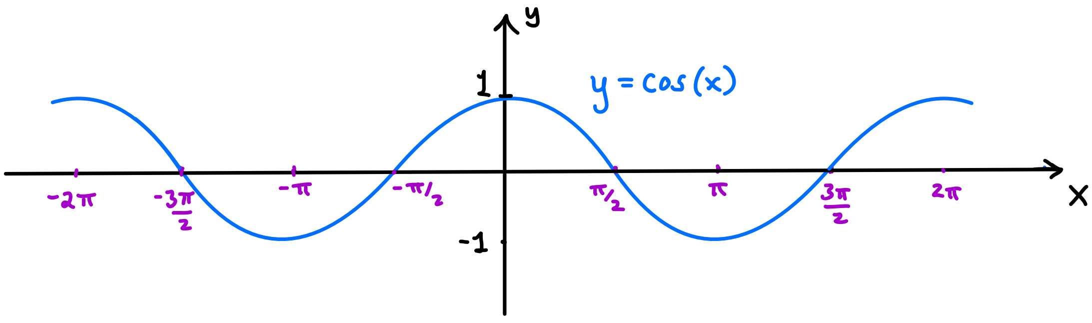
We can use the values of $\cos\theta$ and $\sin\theta$ to compute the values of the other trig functions!

Ex: $$\tan\left(\frac{2\pi}{3}\right) = \frac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right)} = \frac{\sqrt{3}/2}{-1/2} = \boxed{-\sqrt{3}}$$

$$\sec\left(\frac{2\pi}{3}\right) = \frac{1}{\cos\left(\frac{2\pi}{3}\right)} = \frac{1}{\left(-1/2\right)} = \boxed{-2}$$

Graphs of Trig Functions





Observations:

- For all x, $\boxed{-1 \le \sin x \le 1\ ,\quad -1 \le \cos x \le 1}$

- $\sin(x)$ is odd : $\boxed{\sin(-x) = -\sin(x)}$

- $\cos(x)$ is even : $\boxed{\cos(-x) = \cos(x)}$

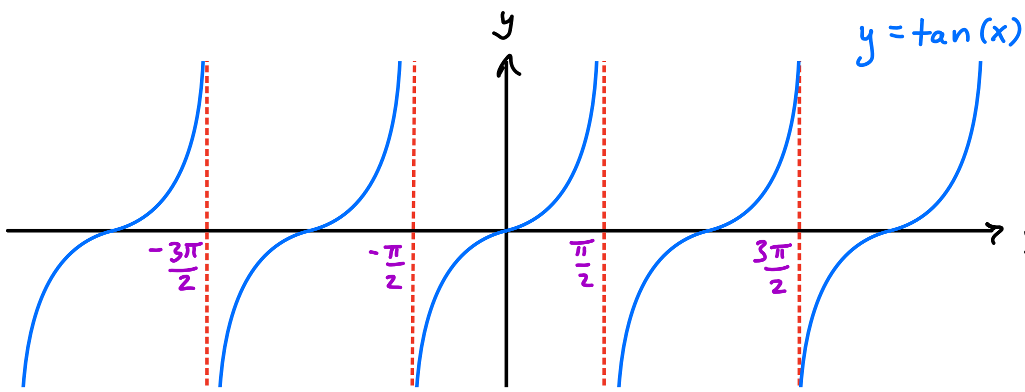
- Both functions are periodic with period 2π :

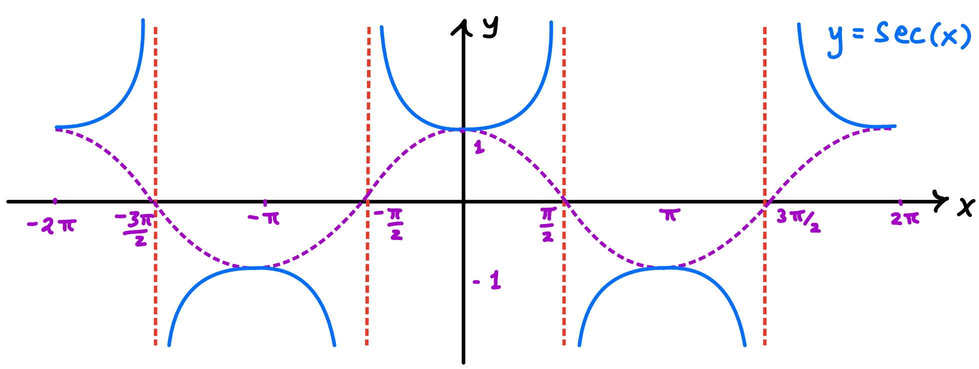
$$\boxed{\sin(x+2\pi) = \sin(x) \ , \ \cos(x+2\pi) = \cos(x)}$$

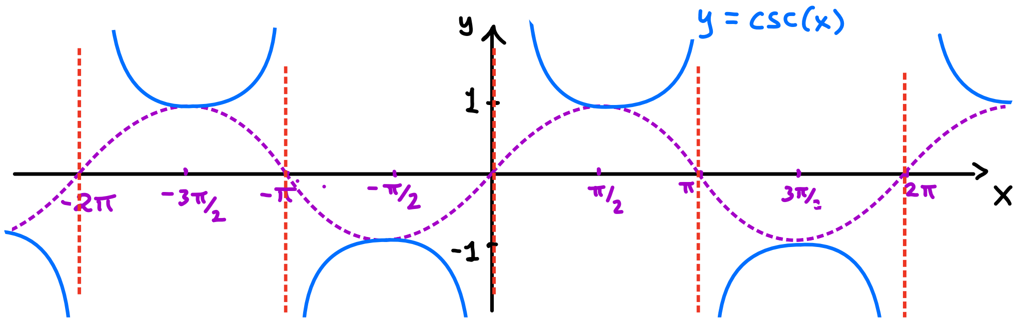
- $\cos(x)$ is a shift of $\sin(x)$ and vice versa :

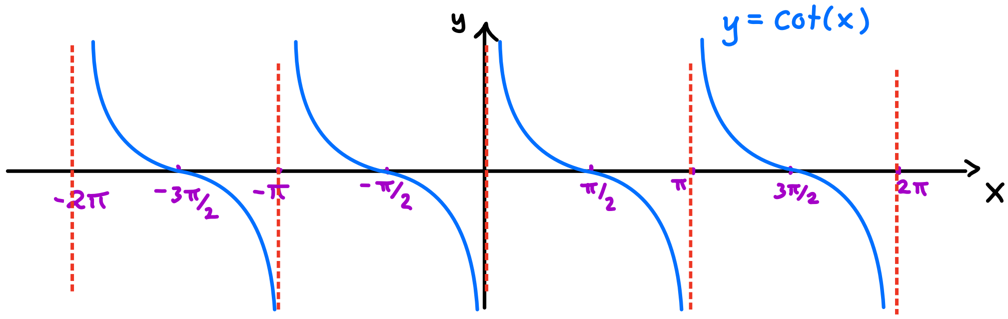
$$\boxed{\cos(x) = \sin(x+\pi/2) \ , \ \sin(x) = \cos(x-\pi/2)}$$

We also have :









Other Useful Identities

Sum / Difference Identities

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta,$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

We can use the above identities to prove new identities!

Ex: Prove that for all θ,

$$2\sin^3\theta = 2\sin\theta - \sin 2\theta \cos\theta.$$

Tip: start with more complicated side!

Solution: Starting with the RHS, we have

$$\begin{aligned} 2\sin\theta - \sin 2\theta\cos\theta &= 2\sin\theta - (2\sin\theta\cos\theta)\cos\theta \\ &= 2\sin\theta\left[1-\cos^2\theta\right] \\ &= 2\sin\theta\left[\sin^2\theta\right] \\ &= 2\sin^3\theta, \end{aligned}$$

which we recognize as the LHS, hence LHS = RHS. ■

Identities are also useful when solving trig equations.

Ex: Find all $\theta \in [0, 2\pi]$ that solve the equation.

(a) $\cos\theta = \sin 2\theta$

Solution:

$$\cos\theta = \sin 2\theta \implies \cos\theta = 2\sin\theta\cos\theta$$

$$\implies \cos\theta(1-2\sin\theta) = 0$$

$$\implies \cos\theta = 0 \quad \text{or} \quad 1-2\sin\theta = 0$$

$$(\Rightarrow \sin\theta = 1/2)$$

From the unit circle

$$\boxed{\theta = \pi/2,\ 3\pi/2,\ \pi/6,\ 5\pi/6}$$

(b) $\sin^2\theta + \cos\theta + 1 = 0$

Solution: Rewrite in terms of $\cos\theta$!

$$\sin^2\theta + \cos\theta + 1 = 0 \implies (1-\cos^2\theta) + \cos\theta + 1 = 0$$

$$\implies -\cos^2\theta + \cos\theta + 2 = 0$$

Looks like

$x^2 - x - 2 = (x-2)(x+1)$

where $x = \cos\theta$.

$$\implies \cos^2\theta - \cos\theta - 2 = 0.$$

$$\implies (\cos\theta - 2)(\cos\theta + 1) = 0$$

$$\implies \cos\theta = 2 \quad \text{or} \quad \cos\theta = -1$$

(impossible!) $\qquad (\theta = \pi)$

$$\therefore \boxed{\theta = \pi}$$