Logarithmic Differentiation

y	x^2	2^x	x^x
y'	$2x$	$2^x \ln(2)$	???

To find the derivative of $y = f(x)^{g(x)}$, we can use logarithmic differentiation.

Idea: Apply ln to both sides and differentiate

Ex: Let $y = x^x$, $x > 0$. Find y'.

Solution: $y = x^x \Rightarrow \ln y = \ln(x^x) = x \ln(x)$

differentiate!

$$\Rightarrow \frac{y'}{y} = (x)' \ln x + x \cdot (\ln x)'$$

$$\Rightarrow y' = y\left(\ln x + x \cdot \frac{1}{x}\right)$$

$$\Rightarrow \boxed{y' = x^x(\ln x + 1)}$$

Ex: Let $y = (\sin x)^{\cos x}$, $\sin x > 0$. Find y'.

Solution: $y = (\sin x)^{\cos x}$

$$\Rightarrow \ln y = \ln\left((\sin x)^{\cos x}\right) = \cos x \cdot \ln(\sin x)$$

differentiate!

$$\Rightarrow \frac{y'}{y} = (\cos x)' \cdot \ln(\sin x) + \cos x \cdot \left[\ln(\sin x)\right]'$$

$$\Rightarrow y' = y\left(-\sin x \cdot \ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x}\right)$$

$$\Rightarrow \boxed{y' = (\sin x)^{\cos x}\left(-\sin x \cdot \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)}$$

We can also use logarithmic differentiation to find tricky derivatives, even when the function isn't $f(x)^{g(x)}$.

Ex: Let $f(x) = \dfrac{x \cdot (x^2+1)^{10} \cdot e^x}{x^4+3}$, $x > 0$.

Use logarithmic differentiation to find $f'(x)$.

Solution:

$$\ln(f(x)) = \ln\left(\frac{x\cdot(x^2+1)^{10}\cdot e^x}{x^4+3}\right)$$

$$= \ln(x) + \ln\left((x^2+1)^{10}\right) + \ln(e^x) - \ln(x^4+3)$$

$$= \ln(x) + 10\ln(x^2+1) + x - \ln(x^4+3)$$

Differentiating, we have

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + 10\cdot\frac{2x}{x^2+1} + 1 - \frac{4x^3}{x^4+3}$$

$$\Rightarrow\ f'(x) = f(x)\left(\frac{1}{x} + \frac{20x}{x^2+1} + 1 - \frac{4x^3}{x^4+3}\right)$$

$$\Rightarrow\ \boxed{f'(x) = \frac{x\cdot(x^2+1)^{10}\cdot e^x}{x^4+3}\left(\frac{1}{x} + \frac{20x}{x^2+1} + 1 - \frac{4x^3}{x^4+3}\right)}$$