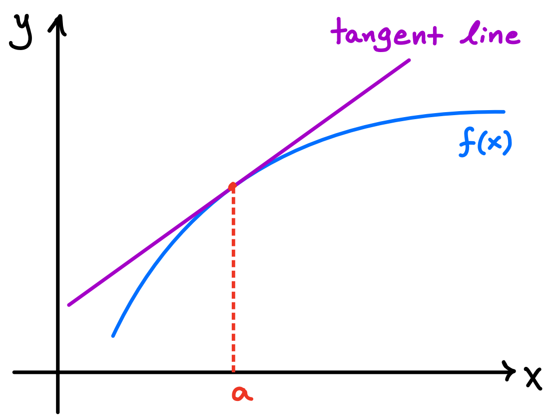
§4.12 - Linear Approximation & Differentials

Recall: If $f'(a)$ exists, then the tangent line to $f(x)$ at $x=a$ is

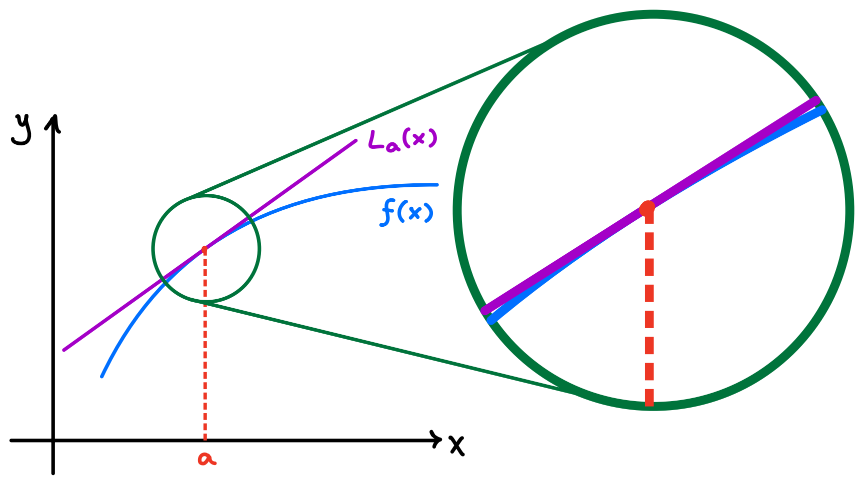
$$\boxed{y = f(a) + f'(x)(x-a)}$$



Let's think of the tangent line as a function, $L_a(x)$.

$$\boxed{L_a(x) = f(a) + f'(a)(x-a)}$$

If we zoom in closely around the point of tangency ...



... we see that $f(x)$ and $L_a(x)$ are nearly indistinguishable! For this reason, $L_a(x)$ is called the linear approximation or the linearization to $f(x)$ at $x=a$.

$$\boxed{f(x) \approx L_a(x) = f(a) + f'(a)(x-a) \quad \text{for } x \text{ near } a.}$$

Ex: Find the linear approximation to $f(x)=\sqrt{x}$ at $x=4$.

Use this to approximate $\sqrt{4.04}$ and $\sqrt{3.92}$.

Solution: $L_4(x) = f(4) + f'(4)(x-4)$ where $f(4) = \sqrt{4} = 2$

and $f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. Hence

$$\boxed{L_4(x) = 2 + \frac{1}{4}(x-4)}$$

$$\begin{aligned}\sqrt{4.04} = f(4.04) \approx L_4(4.04) &= 2 + \frac{1}{4}(4.04-4) \\ &= 2 + \frac{1}{4}(0.04) \\ &= 2 + 0.01 = \boxed{2.01}\end{aligned}$$

Actual value: $\sqrt{4.04} = 2.009975\ldots$ close!!

$$\sqrt{3.92} = f(3.92) \approx L_4(3.92) = 2 + \frac{1}{4}(3.92 - 4)$$

$$= 2 + \frac{1}{4}(-0.08)$$

$$= 2 - 0.02 = \boxed{1.98}$$

Actual Value: $\sqrt{3.92} = 1.979898\ldots$ close!!

Note: If x is far from a, the approximation $L_a(x) \approx f(x)$ is often much less accurate.

From our last example at $x = 100$:

Not close!

$L_4(100) = 2 + \frac{1}{4}(100 - 4) = 26$ while $f(100) = \sqrt{100} = 10$.

Ex: What is the linearization of $f(x) = \sin x$ at $x = 0$?

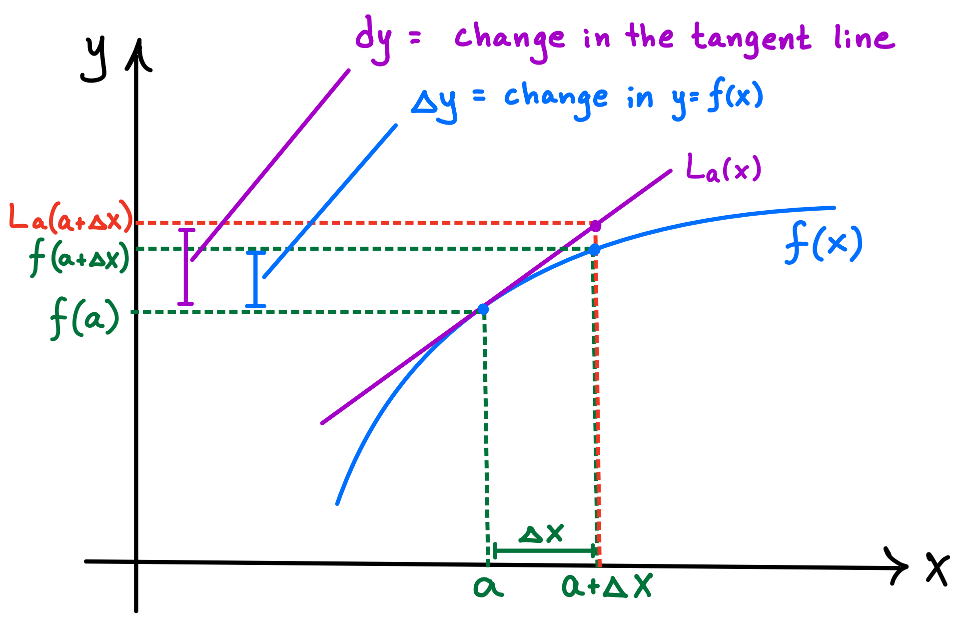
Solution: $f(0) = \sin(0) = 0$ and $f'(0) = \cos(0) = 1$, so

$$L_0(x) = f(0) + f'(0)(x - 0) = x$$

Thus, $\sin(x) \approx x$ when x is close to 0. This small angle approximation is used heavily in Physics & Engineering!

Differentials

We can alternatively think of the tangent line approximation as an estimate of the change in y (Δy) resulting from a small change in x (Δx).



We can calculate the change in the tangent line as

$$dy = L_a(a+\Delta x) - f(a)$$

$$= \left[\cancel{f(a)} + f'(a)\left((\cancel{a}+\Delta x) - \cancel{a}\right) \right] - \cancel{f(a)} \quad = f'(a)\,\Delta x$$

Thus, if Δx is small (in which case we write $\Delta x = dx$), the change in $y = f(x)$, Δy, that we can expect when moving from $x = a$ to $x = a + \Delta x$ is

$$\boxed{\Delta y \approx dy = f'(a)\,dx}$$

We call dx and dy the differentials of x and y, respectively.

Ex: If $f'(1) = 3$, approximate the change in $f(x)$ as we move from $x = 1$ to $x = 1.02$.

Solution: We have $dx = \Delta x = 1.02 - 1 = 0.02$, hence

$$\Delta y \approx dy = f'(1)\,\Delta x = 3 \cdot 0.02 = 0.06.$$

$\therefore$ The y value should increase by ≈ 0.06 when we change the input from $x = 1$ to $x = 1.02$.

Ex: Estimate the change in $y = f(x) = \sqrt{x}$ as x increase from 4 to 4.04.

Solution: We have $dx = \Delta x = 4.04 - 4 = 0.04$ and

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Thus,
$$\begin{aligned} \Delta y \approx dy &= f'(4)\,dx \\ &= \frac{1}{4}(0.04) \\ &= \boxed{0.01.} \end{aligned}$$

Since $f(4) = \sqrt{4} = 2$, this means $f(4.04) \approx 2.01$. This is exactly what we saw in an earlier example!

Additional Exercises

1. Use a linear approximation to estimate $\sqrt[3]{1003}$

2. Estimate the change in volume of a cube when its side length decreases from 10 cm to 9.4 cm.

Solutions:

1. Let's use a linear approximation to $f(x) = \sqrt[3]{x}$ at the nearby point $a = 1000$. We have

$$f(1000) = \sqrt[3]{1000} = 10,$$

$$f'(x) = \left(x^{1/3}\right)' = \frac{1}{3} x^{-2/3} \quad \Rightarrow \quad f'(1000) = \frac{1}{3}(1000)^{-2/3} = \frac{1}{300}$$

hence, the linear approximation is

$$L_{1000}(x) = f(1000) + f'(1000)(x-1000)$$

$$= 10 + \frac{1}{300}(x-1000).$$

Consequently,

$$\sqrt[3]{1003} = f(1003) \approx L_{1000}(1003)$$

$$= 10 + \frac{1}{300}(1003-1000)$$

$$= 10 + \frac{3}{300}$$

$$= \boxed{10.01}$$

2. The volume of the cube is $V = s^3$, where s = side length. We have $V' = 3s^2$, so $V'(10) = 3(10)^2 = 300$.

When s decreases from 10cm to 9.4 cm, we have

$$ds = \Delta s = 10 - 9.4 = -0.6,$$

and therefore the resulting change in volume is

$$\Delta V \approx dV = V'(10)\,ds = 300(-0.6) = \boxed{-180 \text{ cm}^3}$$