Limits at $\pm\infty$

A strategy that sometimes helps to evaluate limits as $x \longrightarrow \pm\infty$ is to factor the largest terms from the numerator and denominator.

Ex: Evaluate $\lim\limits_{x\to\infty} \dfrac{2x^3+x}{7x^3+1}$ ← "$\dfrac{\infty}{\infty}$"

Solution:

$$\lim_{x\to\infty} \frac{2x^3+x}{7x^3+1} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(2+\frac{x}{x^3}\right)}{\cancel{x^3}\left(7+\frac{1}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{2+\cancelto{0}{\frac{1}{x^2}}}{7+\cancelto{0}{\frac{1}{x^3}}} = \frac{2+0}{7+0} = \boxed{\frac{2}{7}}$$

Ex: If $f(x) = \dfrac{e^x+1}{e^{2x}+1}$, find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

Solution: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{e^x+1}{e^{2x}+1}$ ← "$\dfrac{\infty}{\infty}$"

$$= \lim_{x\to\infty} \frac{e^{x}\left(1+\frac{1}{e^x}\right)}{e^{2x}\left(1+\frac{1}{e^{2x}}\right)} \quad = e^{x-2x} = e^{-x} = \frac{1}{e^x}$$

$$= \lim_{x\to\infty} \cancel{\frac{1}{e^x}}^{\,0} \cdot \frac{\left(1+\cancel{\frac{1}{e^x}}^{\,0}\right)}{\left(1+\cancel{\frac{1}{e^{2x}}}^{\,0}\right)}$$

$$= 0 \cdot \frac{1+0}{1+0} = 0 \cdot 1 = \boxed{0}$$

Also $\displaystyle \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\cancel{e^x}^{\,0}+1}{\cancel{e^{2x}}^{\,0}+1} = \frac{0+1}{0+1} = \boxed{1}$

<u>Note:</u> If $\displaystyle\lim_{x\to\infty} f(x) = L$ or $\displaystyle\lim_{x\to-\infty} f(x) = L$ where L is a (finite) real number, then f has a <u>horizontal asymptote</u> (HA) at $y = L$.

<u>Ex:</u> Find all horizontal asymptotes of $f(x) = \dfrac{\sqrt{x^2+1}}{x+3}$

<u>Solution:</u> Let's compute $\displaystyle\lim_{x\to\infty} f(x)$ and $\displaystyle\lim_{x\to-\infty} f(x)$!

"$\frac{\infty}{\infty}$"

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+3} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)} = \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)}$$

Since $x\to\infty$, $|x| = x$.

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}\left(1+\frac{3}{x}\right)}$$

$$= \frac{\sqrt{1+0}}{(1+0)} = 1.$$

Thus, there is a horizontal asymptote at $y=1$.

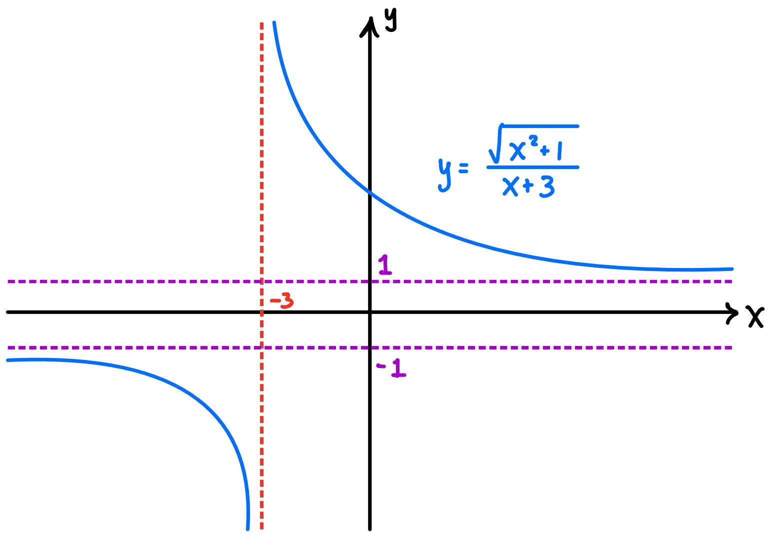
$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+3} \quad \text{"}\frac{\infty}{\infty}\text{"} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)} = \lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)}$$

Since $x \to -\infty$, $|x| = -x$.

$$= \lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}\left(1+\frac{3}{x}\right)}$$

$$= \frac{-\sqrt{1+0}}{(1+0)} = -1$$

Thus, there is a horizontal asymptote at $y=-1$.

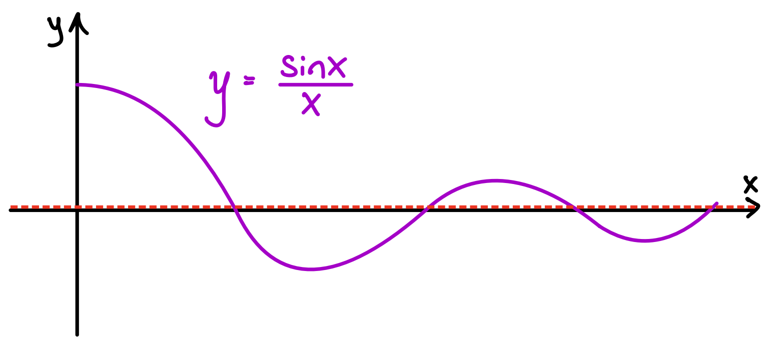


This function has two horizontal asymptotes!

Can a function cross its horizontal asymptote? Yes!

e.g., We showed previously that $\lim_{x \to \infty} \frac{\sin x}{x} = 0$, hence

$y = \frac{\sin x}{x}$ has a HA at $y = 0$.



However, the graph crosses $y=0$ infinitely many times!

Infinite Limits

Limits of the form "$\frac{\pm\infty}{\text{constant}}$" or "$\frac{\text{constant}}{0}$" are not indeterminate — they are $\pm\infty$ depending on signs.

Ex: $\lim_{x \to 3^+} \frac{(x-2)(x-5)}{(x-1)(x-3)} = -\infty$ since the limit has the form "$\frac{1 \cdot (-2)}{2 \cdot 0^+}$" where 0^+ denotes a positive quantity approaching 0.

Note: If a is a (finite) real number and $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$, then f has a vertical asymptote (VA) at $x = a$.

Ex: Does $f(x) = \dfrac{x-2}{x^3-2x^2}$ have any vertical asymptotes?

Solution: Vertical asymptotes may occur when attempting to divide by 0. Note that

$$x^3 - 2x^2 = 0 \iff x^2(x-2) = 0 \iff x = 0 \text{ or } x = 2.$$

Let's now check $\lim_{x \to 0} f(x)$ and $\lim_{x \to 2} f(x)$.

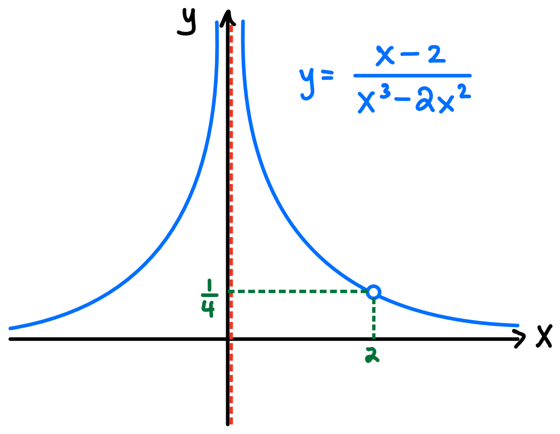
"$\frac{1}{0^+}$"

$$\lim_{x \to 0} \frac{x-2}{x^3-2x^2} = \lim_{x \to 0} \frac{\cancel{x-2}}{x^2\cancel{(x-2)}} = \lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$\Rightarrow \boxed{\text{VA at } x = 0.}$$

$$\lim_{x \to 2} \frac{x-2}{x^3-2x^2} = \lim_{x \to 2} \frac{\cancel{x-2}}{x^2\cancel{(x-2)}} = \lim_{x \to 2} \frac{1}{x^2} = \frac{1}{2^2} = \frac{1}{4}$$

$$\Rightarrow \boxed{\text{No VA at } x = 2.}$$



We see a VA at $x=0$ and a hole at $x=2$.

Additional Exercises:

1. Evaluate $\displaystyle\lim_{x\to-\infty} \frac{x^{2/3}+x^{1/3}}{x^{2/3}+1}$

2. Evaluate $\displaystyle\lim_{x\to\infty} \left(\sqrt{x+2}-\sqrt{x}\right)$

3. Does $f(x)=\dfrac{\sin x}{\tan x}$ have any vertical or horizontal asymptotes?

Solutions:

1. $$\lim_{x\to-\infty} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\frac{x^{2/3}+x^{1/3}}{x^{2/3}+1}} = \lim_{x\to-\infty} \frac{\cancel{x^{2/3}}\left(1+\frac{x^{1/3}}{x^{2/3}}\right)}{\cancel{x^{2/3}}\left(1+\frac{1}{x^{2/3}}\right)}$$

$$= \lim_{x\to-\infty} \frac{\left(1+\cancelto{0}{\frac{1}{x^{1/3}}}\right)}{\left(1+\cancelto{0}{\frac{1}{x^{2/3}}}\right)} = \frac{1+0}{1+0} = \boxed{1}$$

2. $$\lim_{x\to\infty} \underset{\text{"}\infty-\infty\text{"}}{\left(\sqrt{x+2}-\sqrt{x}\right)} = \lim_{x\to\infty} \frac{\left(\sqrt{x+2}-\sqrt{x}\right)}{1} \cdot \frac{\left(\sqrt{x+2}+\sqrt{x}\right)}{\left(\sqrt{x+2}+\sqrt{x}\right)}$$

$$= \lim_{x\to\infty} \frac{(x+2)-x}{\sqrt{x+2}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x+2}+\sqrt{x}} = \boxed{\infty}$$

3. Vertical asymptotes of $f(x) = \frac{\sin x}{\tan x}$ could occur when $\tan x = 0$, or equivalently, when $x = k\cdot\pi$, $k \in \mathbb{Z}$.

However,

$$\lim_{x\to k\pi} \overset{\text{"}\frac{0}{0}\text{"}}{\frac{\sin x}{\tan x}} = \lim_{x\to k\pi} \frac{\cancel{\sin x}}{\left(\frac{\cancel{\sin x}}{\cos x}\right)} = \lim_{x\to k\pi} \cos x = \begin{cases} 1 & \text{if } k \text{ is even,} \\ -1 & \text{if } k \text{ is odd.} \end{cases}$$

Since none of these limits are $\pm\infty$, there are <u>no vertical asymptotes.</u>

For horizontal asymptotes, we examine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. We have

$$\lim_{x\to\pm\infty} \frac{\sin x}{\tan x} = \lim_{x\to\pm\infty} \frac{\cancel{\sin x}}{\left(\frac{\cancel{\sin x}}{\cos x}\right)} = \lim_{x\to\pm\infty} \cos x,$$ which DNE.

Thus, <u>no horizontal asymptotes either!</u>