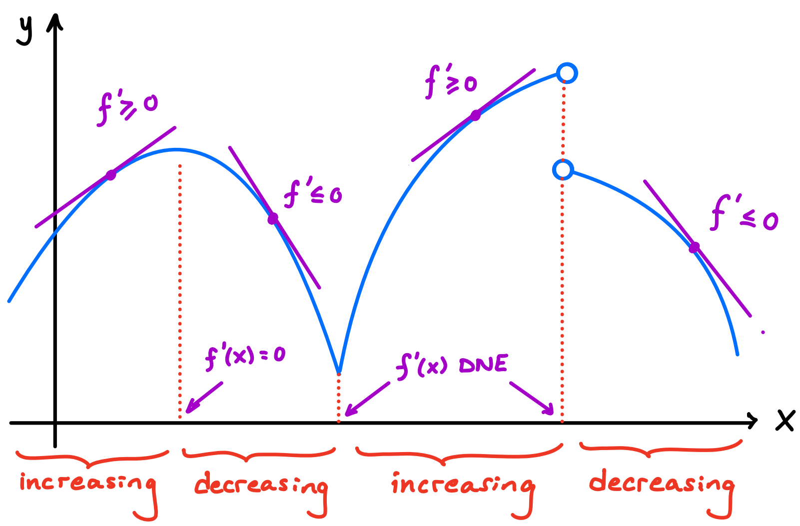
§4.2 – Increasing and Decreasing Functions

In these next few sections, we will learn how to use properties of f' to study the behaviour of f.

In fact, last time we used the MVT to prove the following connection between f and f':

> If $f'(x) \geq 0$ for all $x \in I$, then f is increasing on I.
>
> If $f'(x) \leq 0$ for all $x \in I$, then f is decreasing on I.



Observation:

Intervals of increase/decrease are separated by points where $f'(x) = 0$ or DNE.

This gives us a way to check increasing/decreasing!

Ex: Where is f increasing? Where is f decreasing?

(a) $f(x) = x^2$

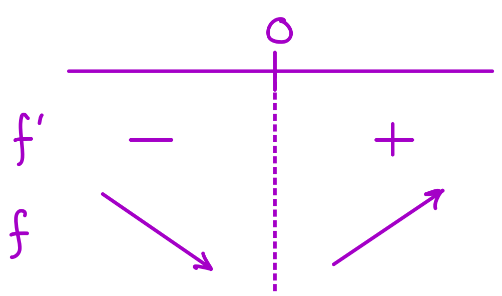
Solution:

[First, find any x's where $f'(x) = 0$ or $f'(x)$ DNE.]

$f'(x) = 2x$, which exists everywhere.

$f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

[Next, check the sign of f' around these points]



$\therefore f$ increasing on $[0, \infty)$

decreasing on $(-\infty, 0]$

(We'll include the endpoint as long as it's in the domain of f.)

(b) $f(x) = x^4 - 4x^3 + 1$

Solution: $f'(x) = 4x^3 - 12x^2$, which exists everywhere.

$$f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0$$

$$\Rightarrow 4x^2(x-3) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 3.$$

	$(-\infty, 0)$	0	$(0, 3)$	3	$(3, \infty)$
f'	$-$		$-$		$+$
f	$\searrow$		$\searrow$		$\nearrow$

$\therefore$ $\boxed{f \text{ is decreasing on } (-\infty, 3] \text{ ; increasing on } [3, \infty).}$

(c) $f(x) = \dfrac{2-x}{(x+1)^2}$

<u>Solution:</u>

$$f'(x) = \frac{(x+1)^2(2-x)' - (2-x)\left[(x+1)^2\right]'}{(x+1)^4}$$

$$= \frac{-(x+1)^{\cancel{2}} - 2(2-x)\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}}$$

$$= \frac{-(x+1) - 2(2-x)}{(x+1)^3} = \frac{x-5}{(x+1)^3}$$

DNE when $x = -1$,

$f'(x) = 0$ when $x = 5$.

	$(-\infty, -1)$	-1	$(-1, 5)$	5	$(5, \infty)$
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$

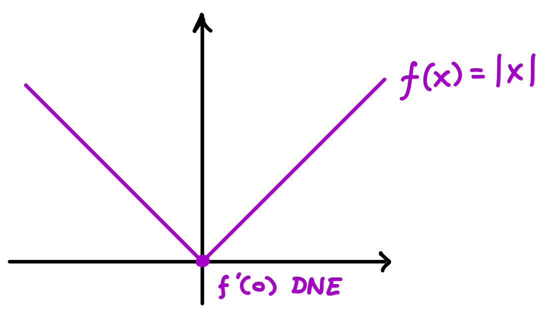
$\therefore$ $\boxed{\begin{array}{l} f \text{ is increasing on } (-\infty,-1) \text{ and on } [5,\infty). \\ f \text{ is decreasing on } (-1,5]. \end{array}}$

(Not included since -1 not in domain of f.)

A point $x=c$ in the domain of f is said to be a critical point (CP) of f if $f'(c)=0$ or $f'(c)$ DNE.

These points will be very important when studying maxima and minima in the next section.

Ex: $f(x)=|x|$ has a CP at $x=0$ since $f'(0)$ DNE.



Ex: We saw that $f(x)=x^4-4x^3+1$ has critical points at $x=0$ and $x=3$, since $f'(0)=0$ and $f'(3)=0$.

Ex: We saw that $f(x) = \frac{2-x}{(x+1)^2}$ has a critical point at $x = 5$.

Note: Even though $f'(-1)$ DNE, $x = -1$ is NOT a critical point as it isn't in the domain of f.