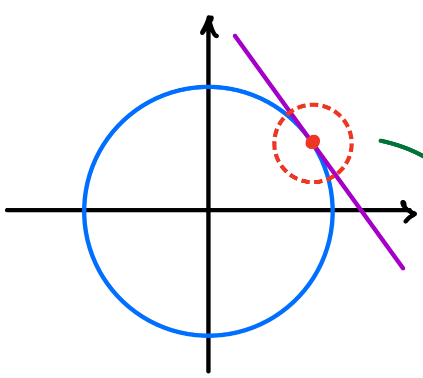
Implicit Differentiation

Up to now, we've worked with explicit functions, $y = f(x)$ (i.e., y alone on the LHS). But sometimes we'll want to find y' when working with something like $x^2 + y^2 = 1$, where the x's and y's are mixed together. We call these implicit functions.

Ex: Let's find the slope of the tangent line to $x^2 + y^2 = 1$ at $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



Around this point, y is a function of x $(y = f(x))$

Idea: Differentiate both sides, treating y as $f(x)$ and applying the chain rule. Then solve for y'.

Solution:

$[f(x)]^2$, derivative is $2f(x)\cdot f'(x)$, or $2y\cdot y'$

$$x^2 + y^2 = 1 \overset{\text{diff.}}{\Longrightarrow} 2x + 2y\cdot y' = 0$$

$$\Longrightarrow 2y\cdot y' = -2x$$

$$\Longrightarrow y' = \frac{-2x}{2y} = \boxed{\frac{-x}{y}}$$

So, at $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ the slope is

$$y' = \frac{-x}{y} = \frac{-\left(\sqrt{2}/2\right)}{\left(\sqrt{2}/2\right)} = \boxed{-1}$$

Ex: Find y' if $2y + y^{23} - x^2 = 1$.

Solution: $2y + y^{23} - x^2 = 1$

$$\overset{\text{diff.}}{\Longrightarrow} 2y' + 23y^{22}\cdot y' - 2x = 0$$

$$\Longrightarrow y'(2 + 23y^{22}) = 2x$$

$$\Longrightarrow \boxed{y' = \frac{2x}{2+23y^{22}}}$$

Ex: Find y' if $3xy^2 + \sin(y) = 4x^3$

Solution: $3xy^2 + \sin(y) = 4x^3$

product rule

$$\stackrel{\text{diff.}}{\Longrightarrow} \left[3y^2 + 3x(2y)\cdot y'\right] + \cos(y)\cdot y' = 12x^2$$

$$\Longrightarrow y'\left(6xy + \cos(y)\right) = 12x^2 - 3y^2$$

$$\Longrightarrow \boxed{y' = \frac{12x^2 - 3y^2}{6xy + \cos(y)}}$$