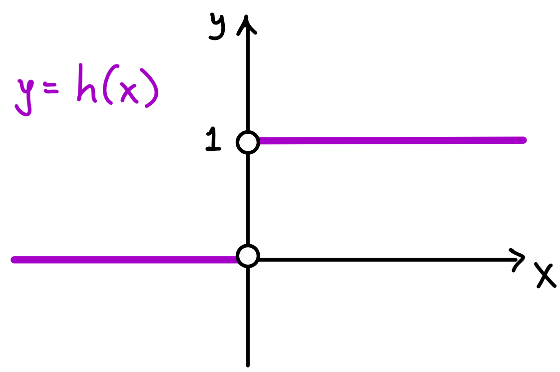
§ 2.5 – Piecewise & Heaviside Functions

The Heaviside step function is defined by

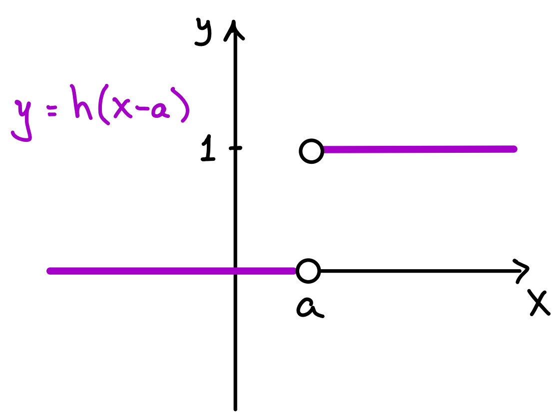
$$\boxed{h(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}}$$

Note: $h(x)$ is not defined at $x = 0$.

$h(x)$ is like a switch that is "off" when $x < 0$ when "turns on" when $x > 0$.



By applying a horizontal shift, we can modify $h(x)$ to "turn on" when $x > a$:

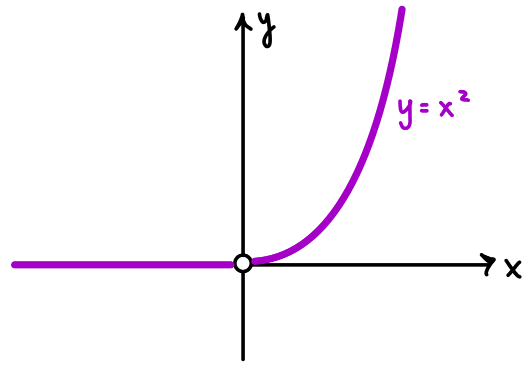


Let's see what else we can model with $h(x)$!

Ex: Sketch the following:

(a) $y = x^2 \cdot h(x)$

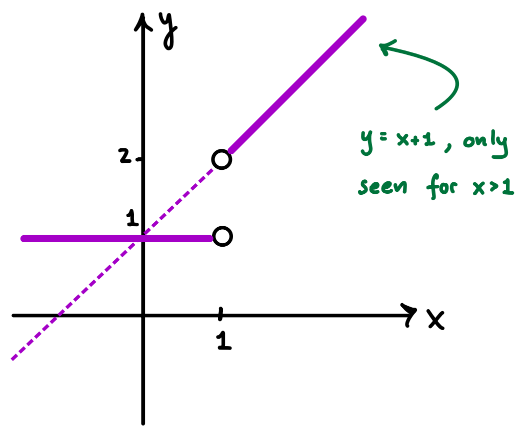
$$= \begin{cases} x^2 \cdot 0 = 0 & \text{if } x < 0, \\ x^2 \cdot 1 = x^2 & \text{if } x > 0. \end{cases}$$



> **Observation 1:** We can "turn on" a function for $x > a$ by multiplying it by $h(x-a)$!

(b) $y = 1 + x \cdot \underbrace{h(x-1)}_{\text{"on" when } x>1}$

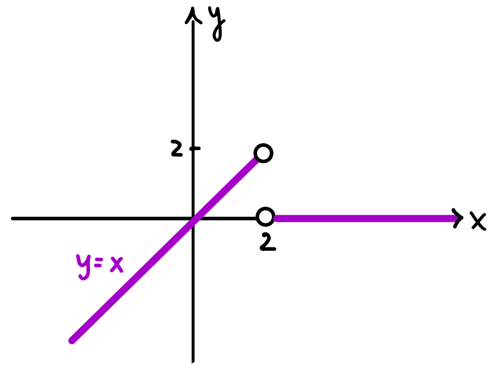
$$= \begin{cases} 1 + x \cdot 0 = 1 & \text{if } x < 1 \\ 1 + x \cdot 1 = 1 + x & \text{if } x > 1 \end{cases}$$



> **Observation 2:** If part of a function has no Heaviside attached, then that part starts "on"

(c) $y = x + (-x)\overbrace{h(x-2)}^{\text{"on" when } x>2}$

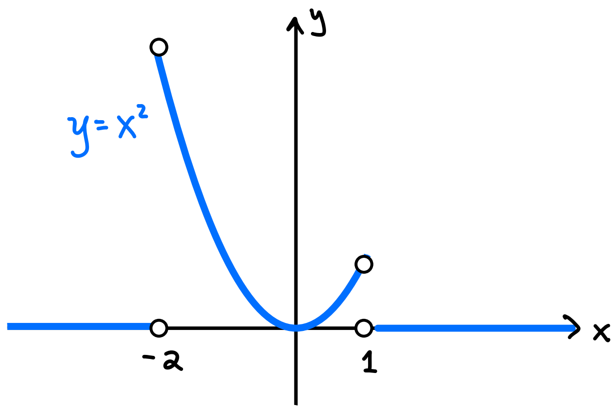
$$= \begin{cases} x + (-x)\cdot 0 = x & \text{if } x<2 \\ x + (-x)\cdot 1 = 0 & \text{if } x>2 \end{cases}$$



Observation 3: We can "turn off" $f(x)$ by adding a Heaviside function that "turns on" $-f(x)$!

Ex: Describe the following in terms of Heaviside functions:

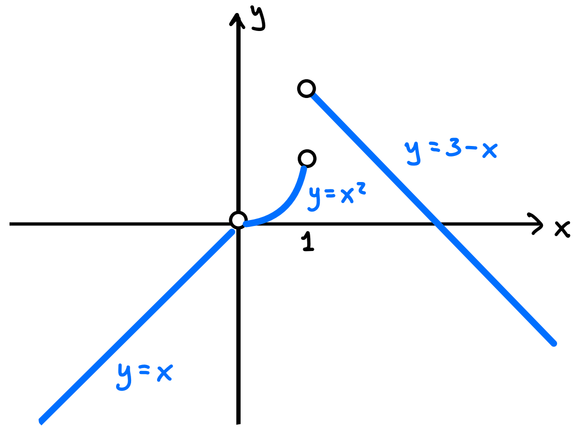
(a)



Solution: $\boxed{y = x^2 \cdot h(x+2) + (-x^2)\cdot h(x-1)}$

$$\underbrace{x^2 \cdot h(x+2)}_{\text{"turn on" } y=x^2 \text{ for } x>-2} + \underbrace{(-x^2)\cdot h(x-1)}_{\text{"turn off" } y=x^2 \text{ for } x>1}$$

(b)



Solution:

y = x starts "on"

$$y = x + \underbrace{x^2 h(x)}_{\text{"turn on" } y=x^2 \text{ for } x>0.} + \underbrace{(-x)\, h(x)}_{\text{"turn off" } y=x \text{ for } x>0.} + \underbrace{(3-x)\cdot h(x-1)}_{\text{"turn on" } y=3-x \text{ for } x>1.} + \underbrace{(-x^2)\, h(x-1)}_{\text{"turn off" } y=-x^2 \text{ for } x>1.}$$

$$\Rightarrow \boxed{y = x + (x^2 - x)\cdot h(x) + (3 - x - x^2)\cdot h(x-1)}$$

(c) $f(x) = \begin{cases} 2x & \text{if } x<3, \\ 1 & \text{if } 3<x<4, \\ x^3 & \text{if } x>4. \end{cases}$

Solution: $\boxed{y = 2x + (1-2x)\cdot h(x-3) + (x^3 - 1)\cdot h(x-4)}$

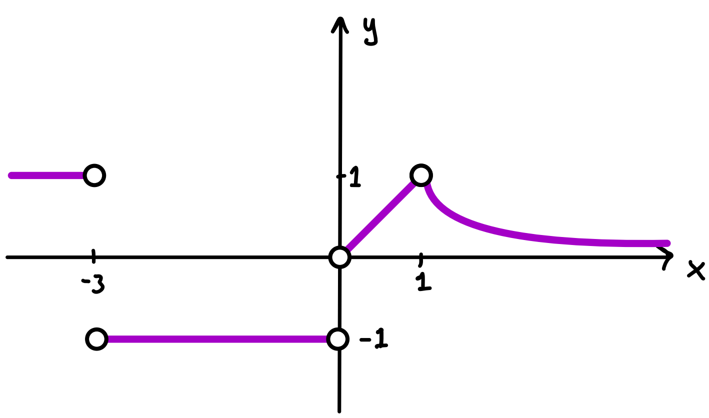
As we can see, the Heaviside function gives us a compact way to describe piecewise functions. We can also write Heaviside functions in piecewise form:

Ex: Write the following as a piecewise function and sketch its graph:

$$f(x) = 1 - 2h(x+3) + (x+1)\cdot h(x) - \left(x - \frac{1}{x}\right)\cdot h(x-1)$$

Solution: Transitions occur around $x = -3, 0, 1$. We have

$$f(x) = \begin{cases} 1 & \text{if } x < -3 \\ 1-2 & \text{if } -3 < x < 0 \\ 1-2+(x+1) & \text{if } 0 < x < 1 \\ 1-2+(x+1)-\left(x-\frac{1}{x}\right) & \text{if } x > 1 \end{cases} = \boxed{\begin{cases} 1 & \text{if } x < -3 \\ -1 & \text{if } -3 < x < 0 \\ x & \text{if } 0 < x < 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}}$$



Additional Exercise:

Sketch the graphs of $y = h(x^2-4)$ and $y = x^2 \cdot h(x^2-4)$.

Hint: When is x^2-4 positive? Negative? Zero?

Solution: We have

$$x^2-4>0 \quad \text{if} \quad x<-2 \text{ or } x>2$$

$$x^2-4<0 \quad \text{if} \quad -2<x<2$$

$$x^2-4=0 \quad \text{if} \quad x=\pm 2,$$

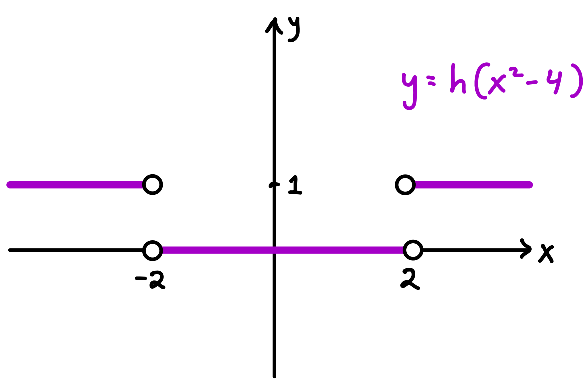
hence

$$h(x^2-4)=1 \quad \text{if} \quad x<-2 \text{ or } x>2$$

$$h(x^2-4)=0 \quad \text{if} \quad -2<x<2$$

$$h(x^2-4) \text{ is undefined if } x=\pm 2.$$

Thus, $h(x^2-4)$ is "on" only when $x<-2$ or $x>2$.



It follows that

$y = x^2 \cdot h(x^2 - 4)$ turns

$y = x^2$ "on" only when

$x < -2$ or $x > 2$.