Generalized Sine Functions

Fact: Every function of the form

angular frequency

$$\boxed{B \cdot \sin(\omega x) + C \cdot \cos(\omega x)}$$

constants that are known to us

can be written in the form

$$\boxed{A \sin(\omega x + \varphi)},$$

constants to be determined

which we call a generalized sine function.

To determine A and φ, let's expand $\sin(\omega x + \varphi)$:

$$\begin{aligned} \boxed{B}\sin(\omega x) + \boxed{C}\cos(\omega x) &= A\sin(\omega x + \varphi) \\ &= A\left[\sin(\omega x)\cos\varphi + \cos(\omega x)\sin\varphi\right] \\ &= \boxed{(A\cos\varphi)}\sin(\omega x) + \boxed{(A\sin\varphi)}\cos(\omega x) \end{aligned}$$

By comparing coefficients on both sides, we get

$B = A\cos\varphi$ and $C = A\sin\varphi$. It then follows that

$$\frac{C}{B} = \frac{\cancel{A}\sin\varphi}{\cancel{A}\cos\varphi} = \tan\varphi,$$

hence $\varphi = \tan^{-1}\left(\frac{C}{B}\right)$. We can then use $B = A\cos\varphi$

(or $C = A\sin\varphi$) to get A.

Summary: To write $B\sin(\omega x) + C\cos(\omega x)$ as $A\sin(\omega x + \varphi)$...

1. Calculate $\varphi = \tan^{-1}\left(\frac{C}{B}\right)$
2. Use $B = A\cos\varphi$ to solve for A.

Ex: Write $2\sin(3x) - 2\cos(3x)$ in the form $A\sin(3x + \varphi)$.

Solution: Here, $B = 2$, $C = -2$. We have

1. $\varphi = \tan^{-1}\left(\frac{C}{B}\right) = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$ and
2. $B = A\cos\varphi \Rightarrow 2 = A\cos\left(-\frac{\pi}{4}\right) = A \cdot \frac{\sqrt{2}}{2} \Rightarrow \frac{4}{\sqrt{2}} = A$

Hence, $\boxed{2\sin(3x) - 2\cos(3x) = \frac{4}{\sqrt{2}}\sin\left(3x - \frac{\pi}{4}\right)}$

Ex: Let $f(x) = 3\sin\left(\frac{x}{2}\right) + 3\sqrt{3}\cos\left(\frac{x}{2}\right)$.

(a) Write $f(x)$ in the form $A\sin\left(\frac{x}{2} + \varphi\right)$.

(b) Find all x such that $f(x) = 6$.

Solution:

(a) Here, $B = 3$, $C = 3\sqrt{3}$. We have

① $\varphi = \tan^{-1}\left(\frac{C}{B}\right) = \tan^{-1}\left(\frac{3\sqrt{3}}{3}\right) = \tan^{-1}(\sqrt{3}) = \underline{\frac{\pi}{3}}$, and

② $B = A\cos\varphi \Rightarrow 3 = A\cos\left(\frac{\pi}{3}\right) = A \cdot \frac{1}{2} \Rightarrow \underline{A = 6}$.

Hence, $\boxed{f(x) = 6\sin\left(\frac{x}{2} + \frac{\pi}{3}\right)}$

(b) It will be easiest to work with the description of $f(x)$ from (a). We have

$$f(x) = 6 \Rightarrow 6\sin\left(\frac{x}{2} + \frac{\pi}{3}\right) = 6$$

$$\Rightarrow \sin\left(\frac{x}{2} + \frac{\pi}{3}\right) = 1$$

$$\Rightarrow \frac{x}{2} + \frac{\pi}{3} = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$$

The set of all integers.

$$\Rightarrow \frac{x}{2} = \left(\frac{\pi}{2} - \frac{\pi}{3}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \frac{x}{2} = \frac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \boxed{x = \frac{\pi}{3} + 4k\pi, \quad k \in \mathbb{Z} \quad \left(= \frac{\pi}{3}, \frac{\pi}{3} \pm 4\pi, \frac{\pi}{3} \pm 8\pi, \ldots\right)}$$