§ 1.5 – Functions

We say that y is a function of x (written $y = f(x)$) if there is a rule that associates exactly one y-value to every x-value.

x is the independent variable

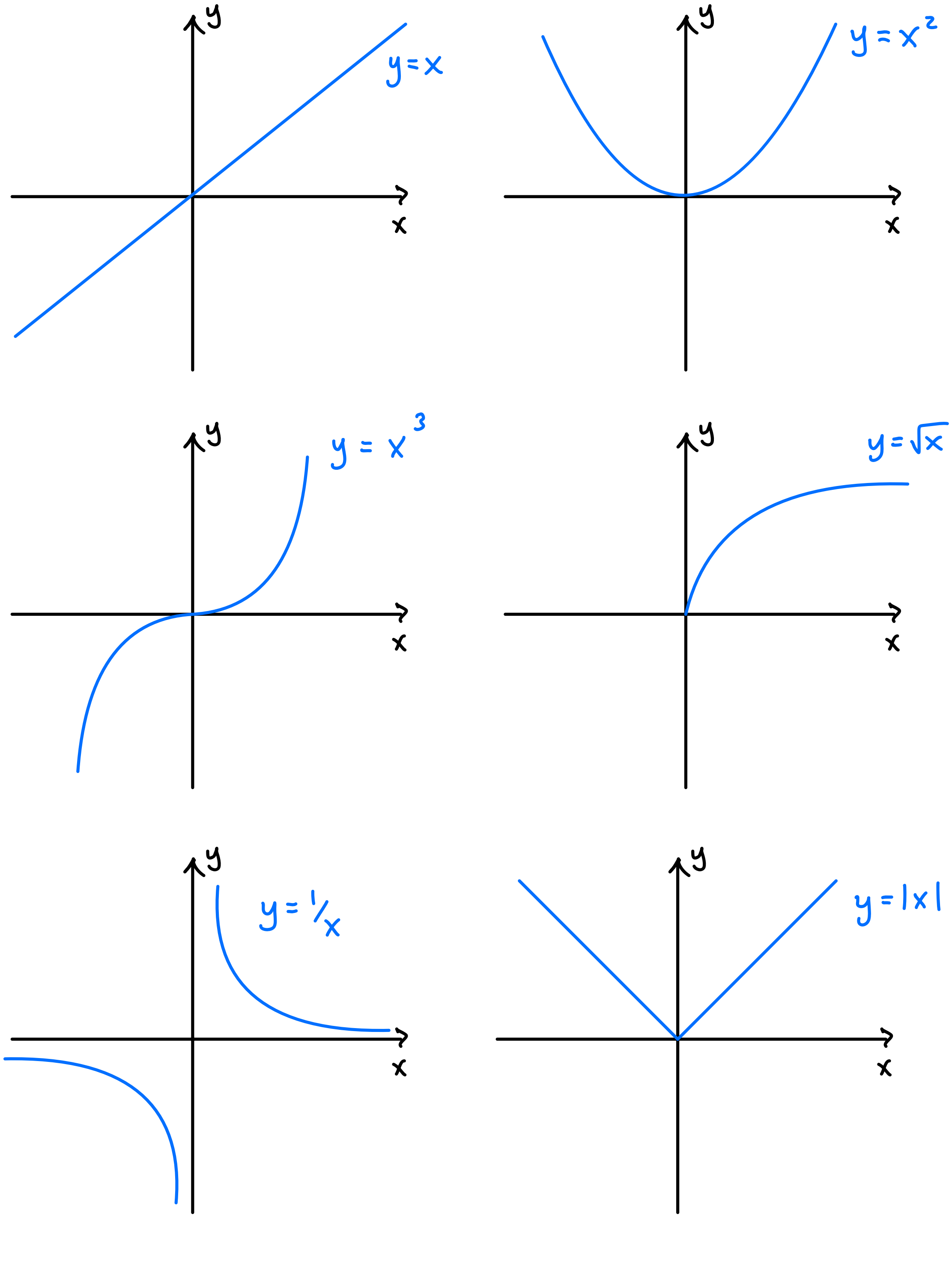
y is the dependent variable

Ex: $y = f(x) = x^3 - x$

e.g., $x = 1 \Rightarrow y = 1^3 - 1 = 0$

$x = -2 \Rightarrow y = (-2)^3 - (-2) = -8 + 2 = -6$

You're probably familiar with the following "basic" functions and their graphs...



and also with the following families of functions:

(i) Polynomials (of degree n)

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}$$

where n is a positive integer, $a_i \in \mathbb{R}$ and $a_n \neq 0$.

The set of all real numbers, also written as $(-\infty, \infty)$.

Ex: $f(x) = 2x^3 + x - 1$ is a polynomial of degree 3.

(ii) Rational functions, which have the form

$$\boxed{f(x) = \frac{P(x)}{Q(x)}}$$ where P, Q are polynomials.

Ex: $f(x) = \dfrac{2x^3 + x - 1}{x^5 + 1/2}$ is a rational function.

Note: Not every equation we encounter will describe a function $y = f(x)$!

Ex: $x = y^2$

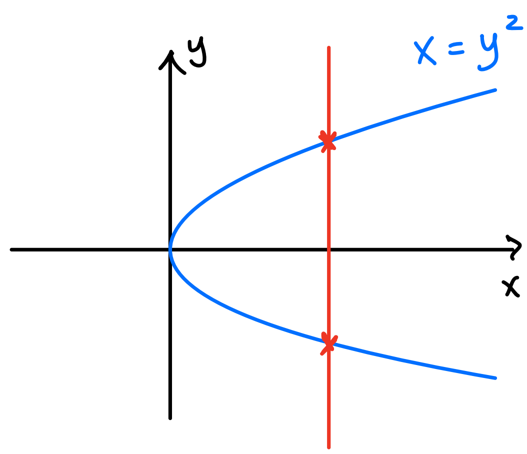
Here y isn't a function of x since most x's produce two y-values: $y = \pm\sqrt{x}$

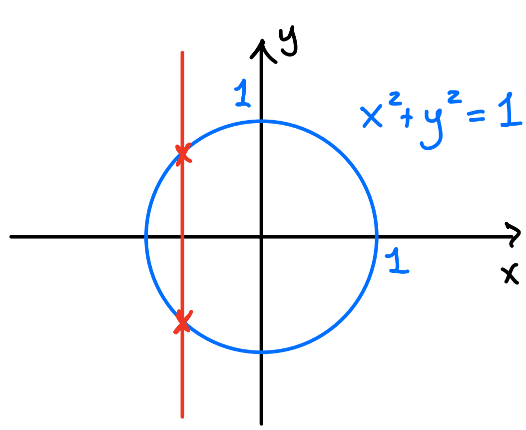
Ex: The unit circle $x^2 + y^2 = 1$

Same issue: $y^2 = 1 - x^2 \Rightarrow y = \pm\sqrt{1 - x^2}$

$\Rightarrow$ Not a function.

We can also see this graphically:





These curves fail the vertical line test and hence aren't graphs of functions $y = f(x)$.

Vertical Line Test: Any vertical line intersects the graph of a function $y = f(x)$ at most once.

You may see the function notation

$$\boxed{f : D \longrightarrow \mathbb{R}.}$$

Here, D is the domain — the set of all inputs x that can be plugged into f.

Possible Domain Issues:

- Division by 0
- Square root (or $\sqrt[n]{}$ where n is even) of something < 0.
- Log of something ≤ 0.

Ex: Find the domain of $f(x) = \sqrt{\dfrac{x-3}{x+2}}$.

Solution: Can't divide by 0, so $x \neq -2$.

Also need $\dfrac{x-3}{x+2} \geq 0$. We can look at the sign of the numerator ($=0$ at $x=3$) and denominator ($=0$ when $x=-2$).

		-2		3	
$x-3$	$-$		$-$		$+$
$x+2$	$-$		$+$		$+$
$\dfrac{x-3}{x+2}$	$+$		$+$		$+$

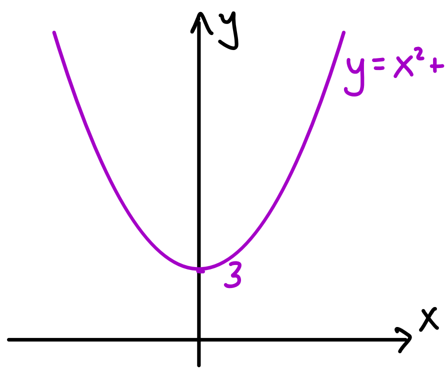
"union"

$\therefore$ Domain is $\boxed{(-\infty, -2) \cup [3, \infty),}$ or equivalently

$$\boxed{\{x \in \mathbb{R} : x < -2 \ \text{ or } \ x \geq 3\}}$$

The range of f is the set of *all* outputs y that f can produce from the x's in its domain.

Ex: $f: \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2 + 3$ has range $[3, \infty)$ since $y = x^2 + 3 \geq 0 + 3 \Rightarrow y \in [3, \infty)$



And every y in $[3, \infty)$ can indeed be produced!

Ex: Find the range of the function

$$f : (1, \infty) \longrightarrow \mathbb{R}, \quad f(x) = \frac{1}{\sqrt{x-1}}.$$

Solution: Let's think about $y = \frac{1}{\sqrt{x-1}}$ for $x \in (1, \infty)$.

- When x is huge, $y = \frac{1}{\sqrt{x-1}}$ is very close to 0 (but $\neq 0$, since numerator $\neq 0$).
- When x is close to 1, $\frac{1}{\sqrt{x-1}}$ blows up to ∞.

$\therefore$ $\boxed{\text{Range} = (0, \infty).}$