§1.9 - Exponential & Logarithmic Functions

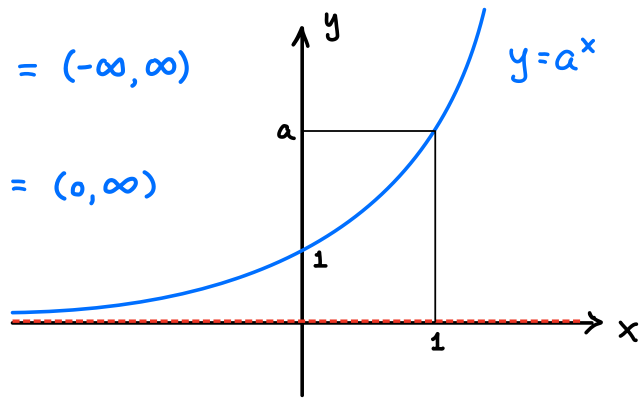
The exponential function $y = a^x$ obeys our usual exponent laws:

$a > 0,\ a \neq 1$

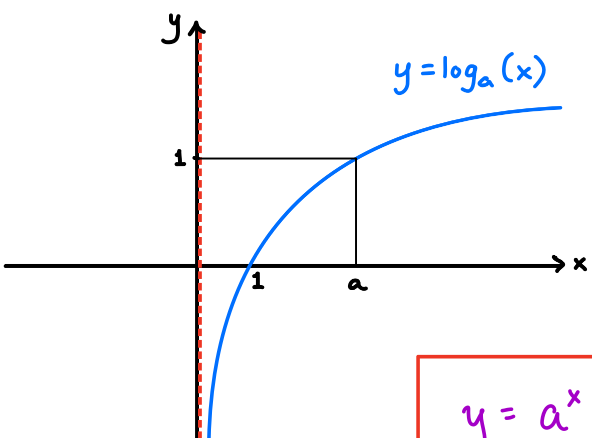
$$\boxed{\begin{aligned} a^{b+c} &= a^b a^c \\ a^{b-c} &= \frac{a^b}{a^c} \\ (a^b)^c &= a^{bc} \end{aligned}}$$

Domain = $(-\infty, \infty)$

Range = $(0, \infty)$



Its inverse is the function $y = \log_a(x)$.



Domain = $(0, \infty)$

Range = $(-\infty, \infty)$

$$\boxed{y = a^x \iff \log_a(y) = x}$$

Ex: $\log_{10}(1000) = 3$ since $10^3 = 1000$.

$\log_2(1/4) = -2$ since $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$.

This function satisfies the following logarithm laws:

$$\boxed{\begin{aligned} \log_a(bc) &= \log_a(b) + \log_a(c) \\ \log_a\left(\frac{b}{c}\right) &= \log_a(b) - \log_a(c) \\ \log_a(b^c) &= c \cdot \log_a(b) \end{aligned}}$$

Ex: Solve the following.

(a) $\log_6(3x) = 2$

Solution: $\log_6(3x) = 2 \implies 6^2 = 3x$

$\implies x = 36/3 = \boxed{12}$

(b) $\log_7(x) + \log_7(x+6) = 1$

Solution: $\log_7(x) + \log_7(x+6) = 1$

$\implies \log_7\left(x(x+6)\right) = 1$

$\implies 7^1 = x(x+6)$

$\Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow (x+7)(x-1) = 0.$

$\Rightarrow x = -7 \quad \text{or} \quad x = 1$

$\log_7(x)$ & $\log_7(x+6)$ are not defined when $x = -7$, but $x = 1$ is okay!

$\therefore x = 1$ is the only solution

Since a^x and $\log_a(x)$ are inverse, it follows that

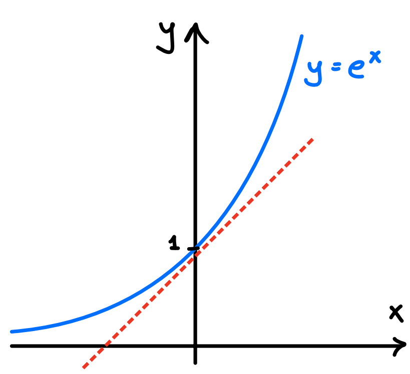
$$\boxed{a^{\log_a(x)} = x \quad , \quad \log_a(a^x) = x}$$

Ex: Simplify $5^{3\log_5(x)}$.

Solution: $5^{3\log_5(x)} = 5^{\log_5(x^3)} = \boxed{x^3}$

$$\boxed{\text{Special Base: } \quad e \approx 2.71828\ldots \quad \text{(Euler's Constant)}}$$

$a = e$ is the unique number such that the graph of $y = a^x$ has slope 1 at $x = 0$.

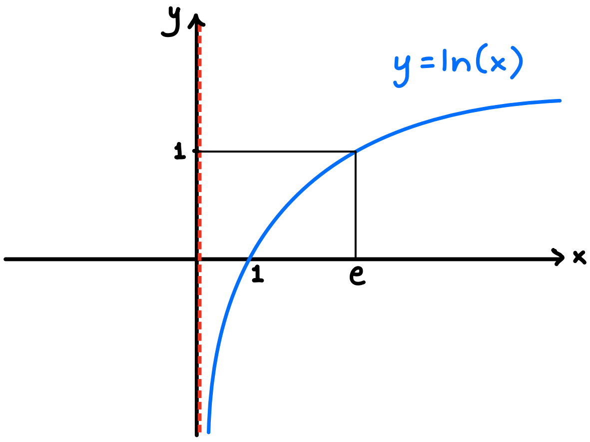


Alternatively, e can be defined by the following famous limit:

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

We call $\log_e(x)$ the natural logarithm and write

$$\boxed{\ln(x) = \log_e(x)}$$



Ex: Solve $\ln(1 + \ln(x)) = 3$

Solution: $\ln(1+\ln(x)) = 3 \Rightarrow e^{\ln(1+\ln(x))} = e^3$

$\Rightarrow 1 + \ln(x) = e^3$

$\Rightarrow \ln(x) = e^3 - 1$

$\Rightarrow e^{\ln(x)} = e^{(e^3-1)}$

$\Rightarrow \boxed{x = e^{(e^3-1)}}$