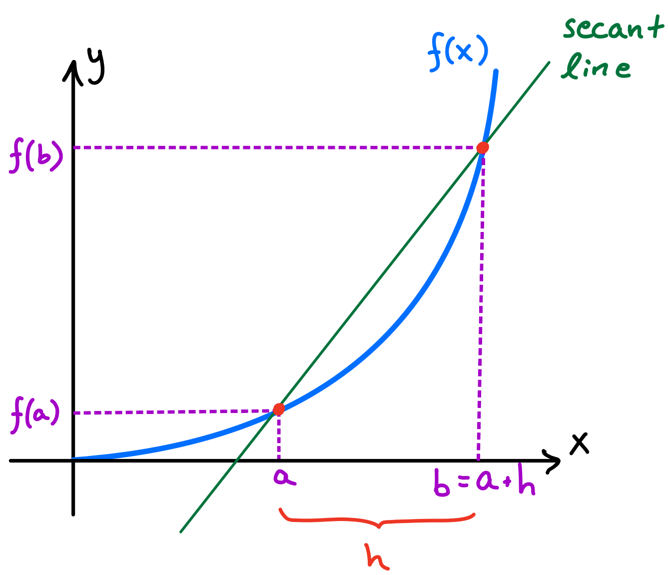
§3.1 – The Derivative

The average rate of change of $f(x)$ from $x = a$ to $x = b$ is $\dfrac{f(b) - f(a)}{b - a}$, the slope of the secant line through $(a, f(a))$ and $(b, f(b))$.



$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{h}$$

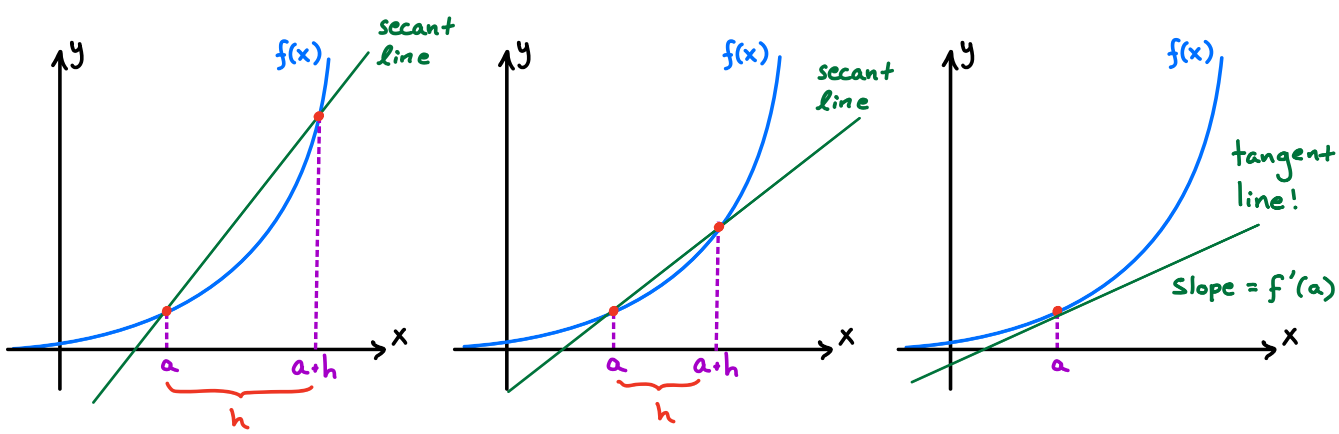
Considering the limit as $h \to 0$, we obtain the instantaneous rate of change at $x = a$:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},}$$

also known as the derivative of f at $x = a$.

> **Note:** One may equivalently define
>
> $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Note that as $h \to 0$ ...



... our secant line becomes a tangent line!

> The tangent line to f at $x = a$ is the line with slope $f'(a)$ and passing through $(a, f(a))$.

Ex: If $f(x) = x^2$, what is $f'(2)$?

Solution: Here, $a = 2$. We have

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{4} + 4h + h^2) - \cancel{4}}{h}$$

$$= \lim_{h \to 0} \frac{h(4+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 4 + \overset{0}{\cancel{h}} = \boxed{4}$$

Rather than repeating this process for various values of a, we can instead find the derivative function

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Also written $\frac{dy}{dx}$ or $\frac{df}{dx}$

and choose the input later.

Ex: For $f(x) = x^2$, the derivative function is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{x^2} + 2xh + h^2) - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + 0 = \boxed{2x}$$

We may now quickly compute, for example,

$f'(2) = 2(2) = 4$, $f'(5) = 2(5) = 10$, etc.

Ex: Find $f'(x)$ if $f(x) = \sqrt{x}$, then find $f'(4)$.

Solution: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\cancel{(x+h)} - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\underbrace{\sqrt{x+h}}_{\to \sqrt{x}} + \sqrt{x}\right)} = \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

Thus, $f'(4) = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$.

<u>Note</u>: We have now seen that

$$(x^2)' = 2x \quad \text{and} \quad (x^{1/2})' = \frac{1}{2} x^{-1/2}$$

In general, we have ...

<u>The Power Rule</u>:

any real number

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

<u>Ex:</u> For $f(x) = x^{4/3}$, we have

$$f'(x) = \frac{4}{3} x^{4/3 - 1} = \frac{4}{3} x^{1/3}$$

Ex: Let $f(x) = |x|$. What is $f'(0)$?

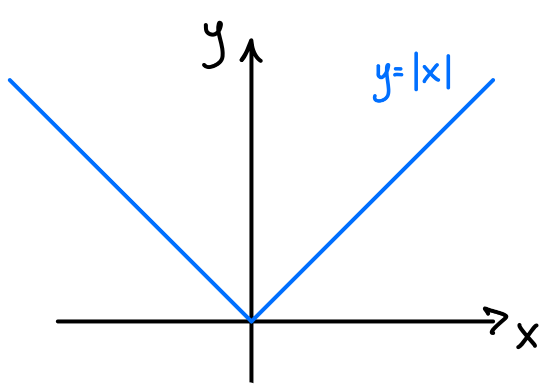
Solution: We have

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

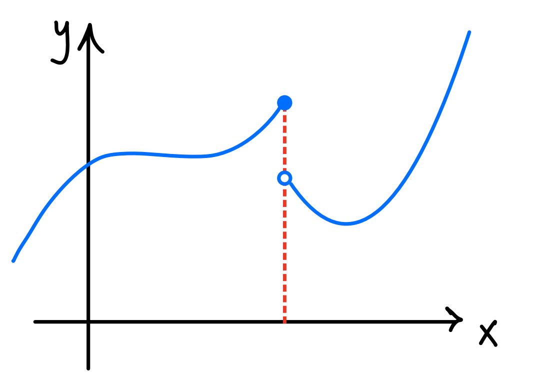
Looking at the one-sided limits, we get

$$\left.\begin{aligned} \lim_{h \to 0^-} \frac{|h|}{h} &= \lim_{h \to 0^-} \frac{-\cancel{h}}{\cancel{h}} = -1 \\ \lim_{h \to 0^+} \frac{|h|}{h} &= \lim_{h \to 0^+} \frac{\cancel{h}}{\cancel{h}} = 1 \end{aligned}\right\} \text{Not equal!}$$

$$\therefore\ f'(0) = \lim_{h \to 0} \frac{|h|}{h} \ \underline{\text{does not exist.}}$$



$f'(a)$ fails to exist at sharp corners (cusps)



$f'(a)$ also fails to exist at points where f is discontinuous

If $f'(a)$ exists, we say f is differentiable at $x = a$.

The above examples tell us two things:

1. If f is differentiable, f must be continuous.

2. Not every continuous function is differentiable.