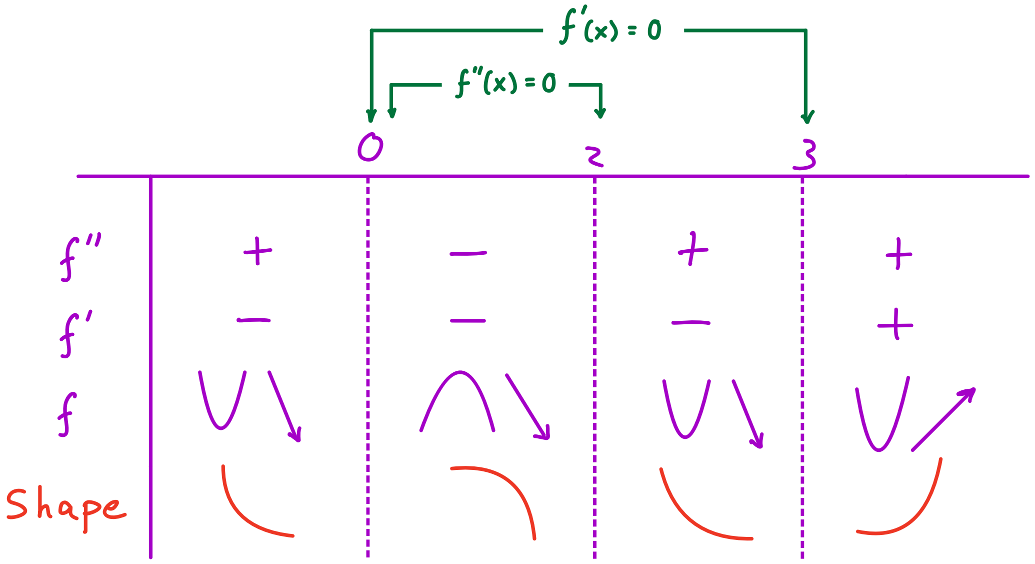
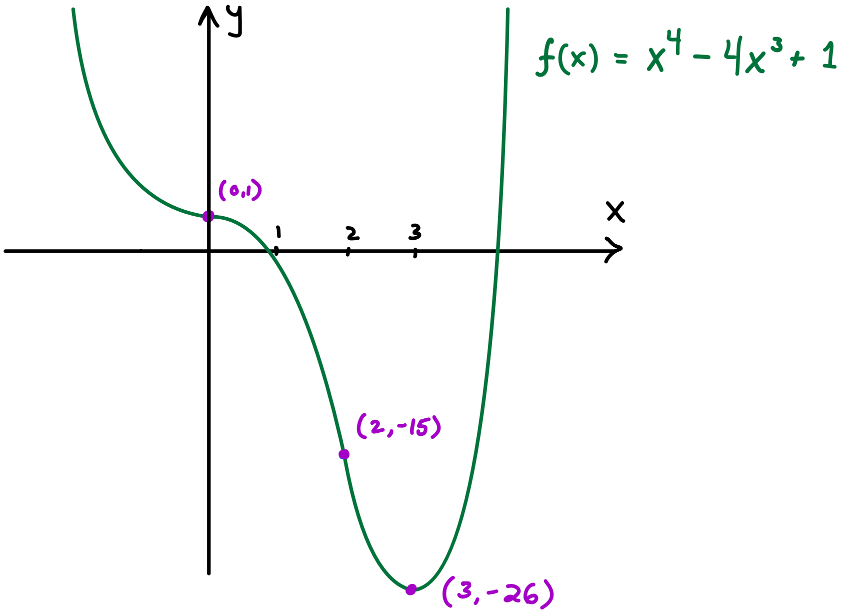
§4.5 – Curve Sketching with Calculus

We've studied the function $f(x) = x^4 - 4x^3 + 1$ in several examples and now have enough information to sketch its graph!



f''	+	−	+	+
f'	−	−	−	+
f	∪ ↘	∩ ↘	∪ ↘	∪ ↗
Shape				

From the table, we see that f has a local min at $(3, f(3)) = (3, -26)$ and inflection points at $(0, f(0)) = (0, 1)$ and $(2, f(2)) = (2, -15)$. We plot these points and connect them using the "Shape" row.



General Strategy: To sketch $y = f(x)$, find

(1) domain of f

(2) vertical asymptotes (any infinite limits?)

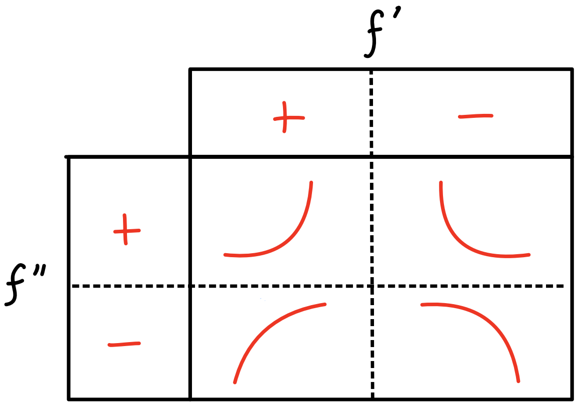
horizontal asymptotes (check limits as $x \to \pm\infty$)

(3) Points where $f'(x) = 0$ or $f'(x)$ DNE

(4) Points where $f''(x) = 0$ or $f''(x)$ DNE.

(5) Test all intervals for increase/decrease and concavity. List local extrema and inflection points.

(6) Plot all interesting points and connect them using the following chart:



Ex: Sketch the graph of $f(x) = e^{-x^2}$

Solution: (1) Domain = $\mathbb{R}$

(2) Vertical asymptotes? None, since domain = $\mathbb{R}$.

Horizontal asymptotes?

$$\lim_{x\to\infty} e^{-x^2} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0 \quad \Longrightarrow \quad \text{HA at } y=0.$$

Similarly, $\lim_{x\to-\infty} e^{-x^2} = 0$.

(3) $\underline{f'(x) = 0 \text{ or DNE?}}$

$$f'(x) = \underbrace{-2xe^{-x^2}}_{\text{exists everywhere}} = 0 \quad \Rightarrow \quad x = 0 \quad (\text{C.P.})$$

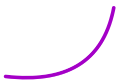
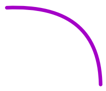
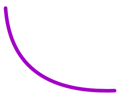
(4) $\underline{f''(x) = 0 \text{ or DNE?}}$

$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = \underbrace{-2(1-2x^2)e^{-x^2}}_{\text{exists everywhere}}$$

$$f''(x) = 0 \quad \Rightarrow \quad 1 - 2x^2 = 0$$

$$\Rightarrow \quad x^2 = 1/2 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

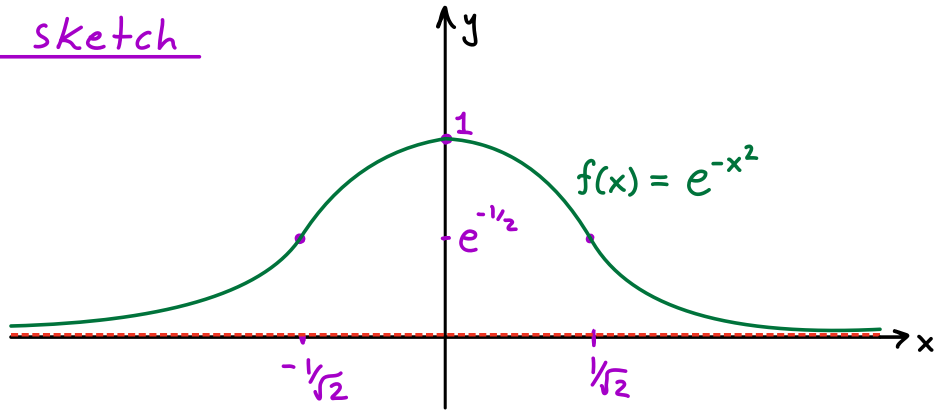
(5)

		$-1/\sqrt{2}$		0		$1/\sqrt{2}$	
f''	$+$		$-$		$-$		$+$
f'	$+$		$+$		$-$		$-$
f	∪ ↗		∩ ↗		∩ ↘		∪ ↘
Shape							

Local max at $(0, f(0)) = (0, 1)$.

Inflection points at $\left(-\frac{1}{\sqrt{2}}, e^{-1/2}\right)$ and $\left(\frac{1}{\sqrt{2}}, e^{-1/2}\right)$

(6) The sketch



Ex: Sketch the graph of $f(x) = \dfrac{x^2}{4 - x^2}$ given that

$$f'(x) = \frac{8x}{(4-x^2)^2} \quad \text{and} \quad f''(x) = \frac{8(3x^2+4)}{(4-x^2)^3}.$$

Solution: (1) $f(x) = \dfrac{x^2}{4-x^2}$ is undefined when $4-x^2=0$, hence when $x = \pm 2$. Thus, domain $= \{x \in \mathbb{R} : x \neq \pm 2\}$

(2) Vertical Asymptotes?

$f(x) = \dfrac{x^2}{4-x^2} = \dfrac{x^2}{(2-x)(2+x)}$ blows up to $\pm\infty$ as $x \longrightarrow \pm 2$.

$\Rightarrow$ VAs at $x = -2$ & $x = 2$.

Horizontal Asymptotes?

$$\lim_{x\to\pm\infty}\frac{x^2}{4-x^2}=\lim_{x\to\pm\infty}\frac{\cancel{x^2}}{\cancel{x^2}\left(\frac{4}{x^2}-1\right)}=\frac{1}{0-1}=-1.$$

$\Rightarrow$ HA at $y=-1$.

(3) $f'(x)=0$ or DNE?

$$f'(x)=\frac{8x}{(4-x^2)^2}$$

$\xrightarrow{=0}$ when $x=0$ (critical pt)

$\xrightarrow{DNE}$ when $x=\pm 2$.

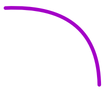
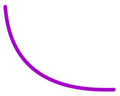
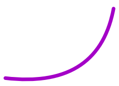
(4) $f''(x)=0$ or DNE?

$$f''(x)=\frac{8(3x^2+4)}{(4-x^2)^3}$$

$\xrightarrow{=0}$ Never, numerator >0.

$\xrightarrow{DNE}$ when $x=\pm 2$.

(5)

		-2		0		2	
f''	$-$		$+$		$+$		$-$
f'	$-$		$-$		$+$		$+$
f	∩ ↘		∪ ↘		∪ ↗		∩ ↗
Shape							

From the table, there is a local min at (0,0);

no inflection points (since $f(\pm 2)$ are undefined)

(6) The Sketch