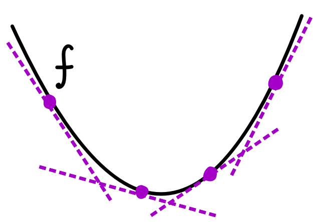
§4.4 – Concavity

We know what $f'(x)$ tells us about $f(x)$, but what information about $f(x)$ can we get from its second derivative, $f''(x)$?

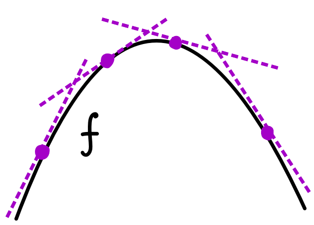
Also written $\frac{d^2f}{dx^2}$ or $\frac{d^2y}{dx^2}$

If $f''(x) \geq 0$ for all x in an interval I, then $f'(x)$ — the slope of the tangent line — is increasing on I.



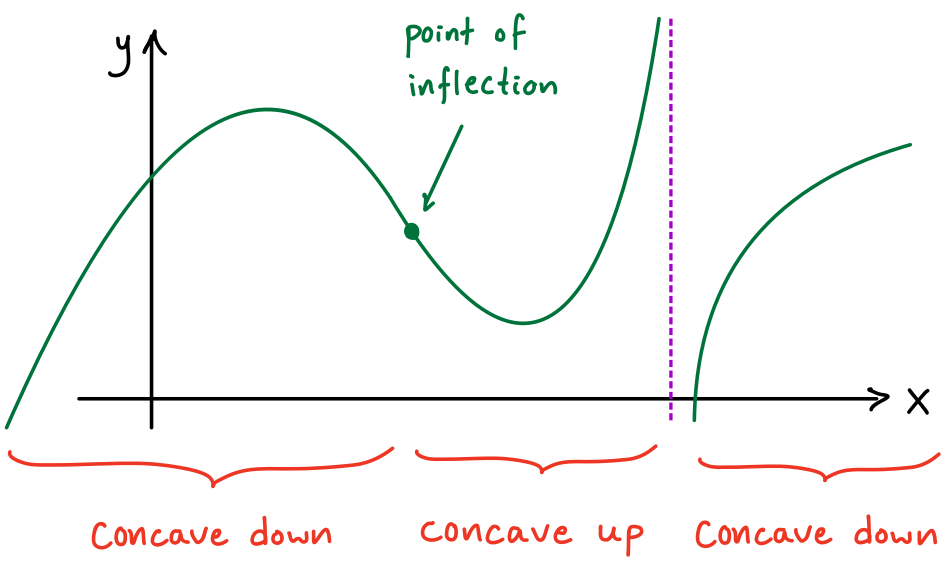
In this case we say f is concave up on I.

If $f''(x) \leq 0$ for all x in an interval I, then $f'(x)$ — the slope of the tangent line — is decreasing on I.



In this case we say f is concave down on I.

Note: Since intervals of increase/decrease of f' are separated by points where its derivative is 0 or DNE, intervals of concavity are separated by points where $f''(x) = 0$ or DNE.



If $x = c$ is a point in the domain of f separating intervals of opposite concavity, we call $x = c$ a point of inflection.

Ex: Find the intervals of concavity and any points of inflection.

(a) $f''(x) = x^2 + 10x - 1$

Solution: $f'(x) = 2x + 10$, $f''(x) = 2$

positive everywhere

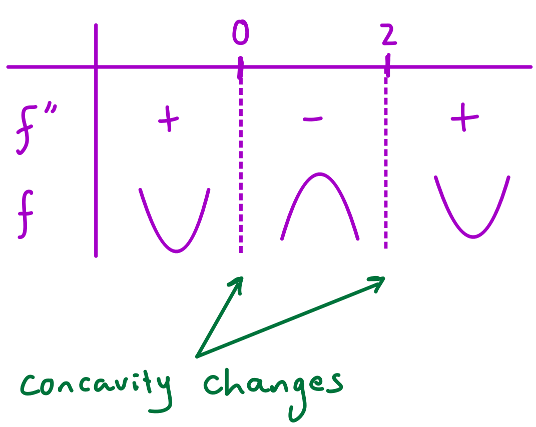
$\therefore$ f is concave up on $(-\infty, \infty)$; no inflection points.

(b) $f(x) = x^4 - 4x^3 + 1$

exists everywhere.

Solution: $f'(x) = 4x^3 - 12x^2$, $f''(x) = 12x^2 - 24x$

$f''(x) = 0 \implies 12x^2(x-2) = 0 \implies x = 0$ or $x = 2$

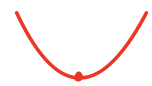


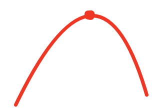
f is concave up on $(-\infty, 0]$ and $[2, \infty)$. f is concave down on $[0, 2]$. There are inflection points at $x = 0$ & $x = 2$.

Concavity gives us a second method for classifying local maxima and minima!

The Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous near $x = c$.

(i) $f''(c) > 0 \Rightarrow$ local min at $x = c$ 

(ii) $f''(c) < 0 \Rightarrow$ local max at $x = c$. 

(iii) $f''(c) = 0 \Rightarrow$ the test gives no information. We may have a local max, local min, or neither.

Ex: Classify the critical point of $f(x) = x^4 - 4x^3 + 1$ at $x = 3$ as a local max, min, or neither.

Solution: $f''(x) = 12x^2 - 24x \Rightarrow f''(3) = 36$.

Since $f''(3) > 0$, there is a $\boxed{\text{local min at } x = 3.}$