§ 6.2 - Diagonalization

In §6.1, we motivated the study of eigenvectors by looking at the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

This diagonal matrix was very easy to work with:

$$A^2 = \begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 2^3 & 0 \\ 0 & 5^3 \end{bmatrix}, \quad \ldots, \quad A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$$

Most other matrices aren't so nice... BUT we can sometimes convert these bad matrices into nice diagonal ones through a process called diagonalization!

Let's see how this process works through the following example:

Ex: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

1. Find the eigenvalues of A and a basis for each eigenspace.

$p_A(\lambda) = (\lambda - 1)(\lambda - 3)$, so eigenvalues are $1, 3$.

For $\lambda = 1$, a basis is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

For $\lambda = 3$, a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

2. Put the eigenvalues in a diagonal matrix D. (order doesn't matter, but repeated eigenvalues should be grouped together)

In our example, take $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

3. Using the same order as in **2.**, put the basis vectors for each eigenspace as columns of a matrix P.

In our example, $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

The basis for the $\lambda = 1$ eigenspace comes first because 1 came first in D!

4. Find P^{-1}! We get $P^{-1}AP = D$ (diagonal!)

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \underset{R_2+R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \underset{R_2 \cdot \frac{1}{2}}{\sim} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right] \underset{\sim}{R_1 - R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right] \qquad \therefore \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We get

$$\underbrace{\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{D}.$$

Ex: Let's try to diagonalize

$$A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

1. $p_A(\lambda) = \det(A - \lambda I)$

$$= \det\left(\begin{bmatrix} 2-\lambda & 2 & 0 \\ -1 & -1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((2-\lambda)(-1-\lambda) + 2\big)$$

$$= (1-\lambda)\left(-2 - 2\lambda + \lambda + \lambda^2 + 2\right)$$

$$= (1-\lambda)(\lambda^2 - \lambda) = -\lambda(1-\lambda)^2$$

The eigenvalues are 0, 1, 1.

Now to find the eigenvectors!

$\underline{\lambda = 0}$:

$$A - 0I = \begin{bmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \overset{R_1 \cdot \frac{1}{2}}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \underset{\substack{R_2 + R_1 \\ R_3 - R_1}}{\sim}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \overset{R_2 \updownarrow R_3}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 - R_2}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

We get $x_3 = t$, $x_2 = -t$, $x_1 = t$

The solution is $\vec{x} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$, So a

basis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda = 1}$:

$$A - 1I = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \underset{\substack{R_2 + R_1 \\ R_3 - R_1}}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

We get $x_3 = t$, $x_2 = s$, $x_1 = -2s$

The solution is $\vec{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $s, t \in \mathbb{R}$.

So a basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. We'll take $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. We'll take $P = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

4. We now find P^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R_2+R_1 \\ R_3-R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right] \underset{R_2(-1)}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right] \overset{R_1+2R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$$

So $P^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$. We get $P^{-1}AP = D$,

i.e., $\begin{bmatrix} -1 & -2 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex: Let's try to diagonalize $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

1. $p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$

The eigenvalues are $\lambda = 3, 3$.

Let's find the eigenvectors!

$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (RREF) $\qquad x_1 = t$, $x_2 = 0$

The solution is $\vec{x} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$, so a basis is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

2. We have $D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

3. We have $P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ... WAT?

How are we supposed to invert this??

We simply didn't get enough eigenvectors from $\lambda = 3$ to make a 2×2 matrix P!

So... we can't diagonalize A! :(

Let's investigate this phenomenon further.

Definition: Let A be an $n \times n$ matrix with eigenvalue λ.

(1) The algebraic multiplicity of λ is the number of times it occurs as a root in the characteristic polynomial of A.

(i.e., the # of times λ is an eigenvalue of A)

(2) The geometric multiplicity of λ is the number of basis vectors we can find in its

corresponding eigenspace.

Ex: With $A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ we have eigenvalues $0, 1, 1$.

So $\lambda = 0$ has algebraic multiplicity 1

$\lambda = 1$ has algebraic multiplicity 2

A basis for the $\lambda = 0$ eigenspace was $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow \lambda = 0$ has geometric multiplicity 1

A basis for the $\lambda = 1$ eigenspace was $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow \lambda = 1$ has geometric multiplicity 2

Ex: From the last example, the eigenvalues of

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ are $3, 3$, so $\lambda = 3$ has algebraic multiplicity 2. Since $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis for its eigenspace, $\lambda = 3$ has geometric multiplicity 1.

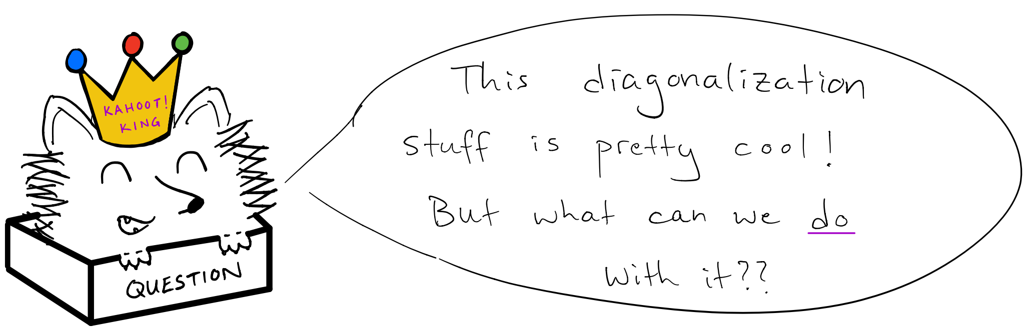
Note: With $A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, each eigenvalue had its alg. mult. equal to its geo. mult. This matrix was diagonalizable (we could write $P^{-1}AP = D$)

With $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, however, the alg. mult. of $\lambda = 3$ was not equal to its geo. mult. This matrix was not diagonalizable (we can't write $P^{-1}AP = D$)

Theorem: Let A be an $n \times n$ matrix. Then A is diagonalizable (we can write $P^{-1}AP = D$) if and only if for each eigenvalue λ

$$\boxed{\text{alg. mult of } \lambda = \text{geo. mult of } \lambda}.$$

Application: Matrix Powers



One important application is taking high powers of matrices!

Suppose A is a diagonalizable $n \times n$ matrix, so we can write $P^{-1}AP = D$ (D diagonal)

Suppose we want to know A^{1000}.

We have $P^{-1}AP = D \Rightarrow \cancel{P(}P^{-1}AP\cancel{)P^{-1}} = PDP^{-1}$

$\Rightarrow A = PDP^{-1}$.

Then $A^{1000} = (PDP^{-1})^{1000}$

$= PD\underbrace{\cancel{P^{-1})(P}}_{=I}D\underbrace{\cancel{P^{-1})(P}}_{=I}D\cancel{P^{-1})} \cdots \cancel{(P}DP^{-1})$

$= PD^{1000}P^{-1}$

In general: $\boxed{A^k = PD^kP^{-1} \quad \text{for } k=1,2,3,\ldots}$

Ex: Consider the matrix $A = \begin{bmatrix} 17 & -15 \\ 20 & -18 \end{bmatrix}$

from §6.1. Compute A^{100}.

Solution: By above, we should first

diagonalize A.

1. $P_A(\lambda) = \begin{vmatrix} 17-\lambda & -15 \\ 20 & -18-\lambda \end{vmatrix}$

$= (17-\lambda)(-18-\lambda) + 300$

$= \lambda^2 + \lambda - 306 + 300$

$= \lambda^2 + \lambda - 6$

$= (\lambda-2)(\lambda+3)$ $\Rightarrow$ Eigenvalues are $2, -3$.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 15 & -15 \\ 20 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ (RREF)

$x_2 = t$
$x_1 = t$

$\Rightarrow$ Solution is $\vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$

A basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

For $\lambda = 3$: $A - (-3)I = \begin{bmatrix} 20 & -15 \\ 20 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$ (RREF)

$x_2 = t$
$x_1 = 3/4\, t$

$\Rightarrow$ The solution is $\vec{x} = t\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$

A basis is $\left\{ \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \right\}$

2. We take $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

3. We take $P = \begin{bmatrix} 1 & 3/4 \\ 1 & 1 \end{bmatrix}$

4. Let's compute P^{-1}:

$$\left[\begin{array}{cc|cc} 1 & 3/4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \underset{R_2 - R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 3/4 & 1 & 0 \\ 0 & 1/4 & -1 & 1 \end{array}\right] \underset{R_2 \cdot 4}{\sim}$$

$$\left[\begin{array}{cc|cc} 1 & 3/4 & 1 & 0 \\ 0 & 1 & -4 & 4 \end{array}\right] \overset{R_1 - 3/4 R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & -4 & 4 \end{array}\right]$$

So $P^{-1} = \begin{bmatrix} 4 & -3 \\ -4 & 4 \end{bmatrix}$ and we get $P^{-1} A P = D$.

Since $A^{100} = P D^{100} P^{-1}$, we have

$$A^{100} = \begin{bmatrix} 1 & 3/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}^{100} \begin{bmatrix} 4 & -3 \\ -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & (-3)^{100} \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{100} & 3/4 \cdot 3^{100} \\ 2^{100} & 3^{100} \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -4 & 4 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4 \cdot 2^{100} - 3^{101} & -3 \cdot 2^{100} + 3^{101} \\ 4 \cdot 2^{100} - 4 \cdot 3^{100} & -3 \cdot 2^{100} + 4 \cdot 3^{100} \end{bmatrix}}$$