Chapter 6 – Eigenvalues & Eigenvectors

§6.1 – Introduction to Eigenvalues & Eigenvectors

Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 17 & -15 \\ 20 & -18 \end{bmatrix}.$$

Which matrix is "nicer"?

The question is vague ... but the answer is still most certainly A. But why?

Well ... let's see how these matrices transform a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

$$\boxed{A\vec{x} = \begin{bmatrix} 2x_1 \\ 5x_2 \end{bmatrix}, \quad A^2\vec{x} = \begin{bmatrix} 4x_1 \\ 25x_2 \end{bmatrix}, \quad A^3\vec{x} = \begin{bmatrix} 8x_1 \\ 125x_2 \end{bmatrix}, \quad \text{etc...}}$$

Multiplying by A is predictable! In fact you could probably even guess that $A^{100}\vec{x} = \begin{bmatrix} 2^{100}x_1 \\ 5^{100}x_2 \end{bmatrix}$

What about B?

$$\boxed{B\vec{x} = \begin{bmatrix} 17x_1 - 15x_2 \\ 20x_1 - 18x_2 \end{bmatrix}, \quad B^2\vec{x} = \begin{bmatrix} -11x_1 + 15x_2 \\ -20x_1 + 24x_2 \end{bmatrix}, \quad B^3\vec{x} = \begin{bmatrix} 113x_1 - 105x_2 \\ 140x_1 - 132x_2 \end{bmatrix}, \text{ etc.}}$$

Much worse! Multiplying by B is seemingly less predictable.

Multiplying by A was okay because A merely <u>stretches</u> the standard basis vectors, but <u>doesn't rotate them</u>!

While B may apply weird transformations to the standard basis, there are <u>OTHER vectors</u> out there that <u>B only stretches</u>!

Ex:

$$B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 & -15 \\ 20 & -18 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 & -15 \\ 20 & -18 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ -12 \end{bmatrix} = -3\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

<u>So B stretches $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ by 2, and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ by -3.</u>

These special vectors are called B's eigenvectors.

The stretching factors (2 & −3) are the corresponding eigenvalues.

Definition: Let A be an $n \times n$ matrix. If $\vec{x}$ is a **non-zero** vector in $\mathbb{R}^n$ such that

$$\boxed{A\vec{x} = \lambda \vec{x}}$$

for some real number λ, then $\vec{x}$ is called an eigenvector for A. The scalar λ is its corresponding eigenvalue.

Ex: Consider once again the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 17 & -15 \\ 20 & -18 \end{bmatrix}$$

- Since $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we have that

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 2$.

- Since $A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix} = 5\begin{bmatrix}0\\1\end{bmatrix}$ we have that

$\begin{bmatrix}0\\1\end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 5$.

What about for B?

- Since $B\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix}$ we have that

$\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of B with eigenvalue $\lambda = 2$.

- Since $B\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}-9\\-12\end{bmatrix} = (-3)\begin{bmatrix}3\\4\end{bmatrix}$ we have that

$\begin{bmatrix}3\\4\end{bmatrix}$ is an eigenvector of B with eigenvalue $\lambda = -3$.

Ex: Let $A = \begin{bmatrix}-3 & 6\\-2 & 4\end{bmatrix}$. Which of the vectors

$\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ are eigenvectors for A?

What are the corresponding eigenvalues?

Solution:

- $A\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}-3 & 6\\-2 & 4\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$

  $\Rightarrow$ $\begin{bmatrix}3\\2\end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.

- Eigenvectors are non-zero, so $\begin{bmatrix}0\\0\end{bmatrix}$ is NOT an eigenvector

- $A\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-3 & 6\\-2 & 4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

  $\Rightarrow$ $\begin{bmatrix}2\\1\end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 0$.

- $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-3 & 6\\-2 & 4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$ $\leftarrow$ Not a multiple of $\begin{bmatrix}1\\1\end{bmatrix}$.

  $\Rightarrow$ $\begin{bmatrix}1\\1\end{bmatrix}$ is NOT an eigenvector of A.

Finding Eigenvalues & Eigenvectors

Suppose A is an $n \times n$ matrix.

A scalar λ is an eigenvalue of A if and only if $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$.

That is, if and only if $A\vec{x} - \lambda\vec{x} = \vec{0}$, or equivalently, $(A - \lambda I)\vec{x} = \vec{0}$.

This means $A - \lambda I$ has a non-zero vector in its nullspace, so by our BIG THEOREM

$$\det(A - \lambda I) = 0.$$

Thus, the eigenvalues of A are the scalars λ such that $\det(A - \lambda I) = 0$, and the corresponding eigenvectors are the vectors in the nullspace of $A - \lambda I$.

Thus, we can find the eigenvalues and eigenvectors of A by following these 3 steps

1. Compute $\det(A - \lambda I)$.

   [This will be some function of λ.]

2. Factor the function from 1 to find where it's 0.

   [These will be our eigenvalues.]

3. For each λ found in 2, solve the system $(A - \lambda I)\vec{x} = \vec{0}$ to find $\mathrm{Null}(A - \lambda I)$.

   [The solutions will be our eigenvectors.]

Ex: Find the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution: Follow the steps above.

(1) $$\det(A-\lambda I) = \det\left(\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right)$$

$$= (3-\lambda)(1-\lambda)$$

(2) $$\det(A-\lambda I) = 0 \iff (3-\lambda)(1-\lambda) = 0$$

$$\iff \boxed{\lambda = 3 \text{ or } \lambda = 1}$$

(These are our eigenvalues)

(3) We find the corresponding eigenvectors by solving the system $(A-\lambda I)\vec{x} = \vec{0}$.

For $\lambda = 3$: $$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \underset{R_2+2R_1}{\sim} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ (RREF)}$$

Then $x_1 = t$, $x_2 = 0$, so the solution is

$$\vec{x} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

(These are our eigenvectors for $\lambda = 3$)

For $\lambda = 1$: $\quad A - 1I = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \overset{R_1 \cdot (\frac{1}{2})}{\sim} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ (RREF)

Then $x_2 = t$, $x_1 = -\frac{1}{2}t$, so the solution is

$$\vec{x} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(These are our eigenvectors for $\lambda = 1$)

Indeed, we can check that

$$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = 1\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Ex: Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

1. $\det(A-\lambda I) = \det\left(\begin{bmatrix} -1-\lambda & -1 & -1 \\ 0 & -2-\lambda & -1 \\ 0 & 2 & 1-\lambda \end{bmatrix}\right)$

expand about 1^{st} col.

$$= (-1-\lambda)\begin{vmatrix} -2-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= -(1+\lambda)\left((-2-\lambda)(1-\lambda)+2\right)$$

$$= -(1+\lambda)\left(\cancel{-2}+2\lambda-\lambda+\lambda^2\cancel{+2}\right)$$

$$= -(1+\lambda)(\lambda+\lambda^2)$$

$$= -\lambda(1+\lambda)(1+\lambda) = -\lambda(1+\lambda)^2$$

2. $\det(A-\lambda I) = 0 \iff -\lambda(1+\lambda)^2 = 0$

$$\iff \boxed{\lambda = 0 \quad \text{or} \quad \lambda = -1.}$$

(These are our eigenvalues)

3. We find the corresponding eigenvectors by solving the system $(A-\lambda I)\vec{x} = \vec{0}$.

For $\lambda = 0$: $\quad A - 0I = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \underset{R_3+R_2}{\overset{R_1\cdot(-1)}{\sim}}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \underset{R_2\left(\frac{-1}{2}\right)}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1-R_2}{\sim} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

We have $x_3 = t$, $x_2 = \frac{1}{2}t$, $x_1 = -3/2\, t$, so the solution is

$$\vec{x} = t \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(These are our eigenvectors for $\lambda = 0$)

For $\lambda = -1$: $\quad A - (-1)I = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \underset{R_3+2R_1}{\overset{\sim}{R_2-R_1}}$

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1(-1)}{\sim} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

We have $x_1 = s$, $x_3 = t$, and $x_2 = -t$

Thus, the solution is

$$\vec{x} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\-1\\1\end{bmatrix}, \qquad s,t \in \mathbb{R}$$

(These are our eigenvectors for $\lambda = -1$)

Remarks:

1. If A is an $n \times n$ matrix, then $\det(A-\lambda I)$ will be a polynomial in λ of degree n (meaning the highest power is λ^n)

   This is called the characteristic polynomial of A. and is sometimes written as $P_A(\lambda) = \det(A-\lambda I)$

2. You may have noticed that we don't ever get just one eigenvector for λ, we get a whole subspace of eigenvectors!

The subspace $\text{Null}(A-\lambda I)$ is called the eigenspace for the eigenvalue λ (the space of all its eigenvectors)

③ Since the eigenspace contains infinitely many eigenvectors, I will often ask for a basis for the eigenspace.

You can find this by using the methods from §3.4:

- Find the general solution to $(A-\lambda I)\vec{x}=\vec{0}$
- Use the vectors on each parameter in your basis.

Ex: In the last example, $\left\{\begin{bmatrix}-3/2\\1/2\\1\end{bmatrix}\right\}$ is a basis for the $\lambda=0$ eigenspace, while $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$ is a basis for the $\lambda=-1$ eigenspace.

Ex: Determine the eigenvalues for $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

For each eigenvalue λ, find a basis for its corresponding eigenspace.

1. The characteristic polynomial is

$$P_A(\lambda) = \det(A - \lambda I)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{bmatrix}\right)$$

Let's expand about this row.

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\Big((1-\lambda)(3-\lambda) + 1\Big)$$

$$= (1-\lambda)\left(3 - 4\lambda + \lambda^2 + 1\right)$$

$$= (1-\lambda)(\lambda^2 - 4\lambda + 4) = (1-\lambda)(\lambda-2)^2$$

② $P_A(\lambda) = 0 \iff (1-\lambda)(\lambda-2)^2 = 0$

$$\iff \boxed{\lambda = 1 \quad \text{or} \quad \lambda = 2}$$

③ We'll find a basis vector for each eigenspace by first computing $\text{Null}(A - \lambda I)$.

For $\lambda = 1$: $A - 1I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \underset{}{\overset{R_1 \updownarrow R_3}{\sim}}$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \underset{R_2 \updownarrow R_3}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \underset{R_2(-1)}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 - 2R_2}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

We have that $x_1 = x_3 = 0$
$x_2 = t$.

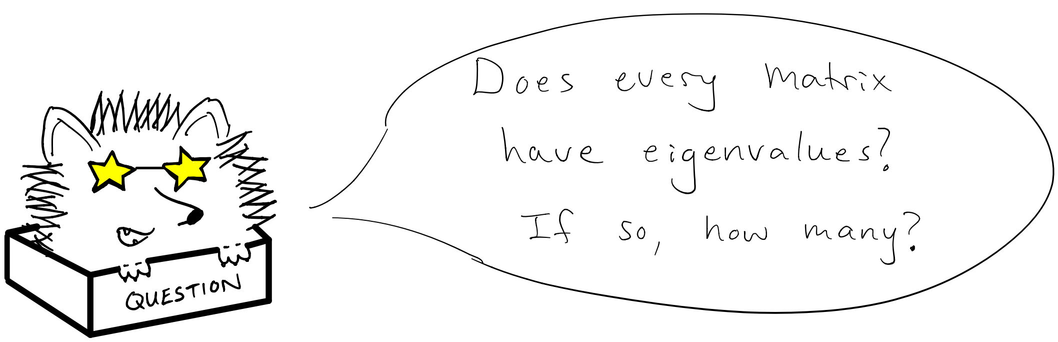
Thus, the solution is $\vec{x} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so a basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

For $\lambda = 2$: $A - 2I = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \underset{\substack{R_2(-1) \\ R_3 + R_1}}{\sim}$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{R_1(-1)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)} \qquad \begin{aligned} x_3 &= t \\ x_2 &= 0 \\ x_1 &= -t \end{aligned}$$

Thus, the solution is $\vec{x} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

$\therefore$ A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.



Ex: $[R_{\pi/2}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalues!

This is because $R_{\pi/2}$ rotates every vector counterclockwise by $\pi/2$, so no vector is merely stretched. Indeed,

$$p_A(\lambda) = \lambda^2 + 1 \implies \text{no real roots!}$$

It does, however, have 2 complex roots: $\pm i$

So $[R_{\pi/2}]$ has no real eigenvalues, but it does have 2 complex ones. In fact,

every $n \times n$ matrix has exactly n eigenvalues (counting repetition), though some may be complex!

Some Useful Eigenvalue Facts

The eigenvalues of an $n \times n$ matrix tell you a lot about its other properties!

Theorem [Eigenvalue Facts]:

Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

1. $\lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$

2. $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ = sum of diagonal entries
$= a_{11} + a_{22} + \cdots + a_{nn}$

Ex: If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then $\det(A) = (2)(2) - (1)(1) = 3$.

and the sum of the diagonal entries is $2+2=4$.

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$ (check)

Sure enough, $\lambda_1 \lambda_2 = 3$ and $\lambda_1 + \lambda_2 = 4$!

From 1. and our BIG THEOREM, we get

A is invertible $\iff \det(A) \neq 0$
$\iff \lambda_1 \lambda_2 \cdots \lambda_n \neq 0$
$\iff$ 0 is not an eigenvalue of A.

This means we can add the statement

$$\boxed{\text{"0 is NOT an eigenvalue of } A\text{"}}$$

to our BIG THEOREM!