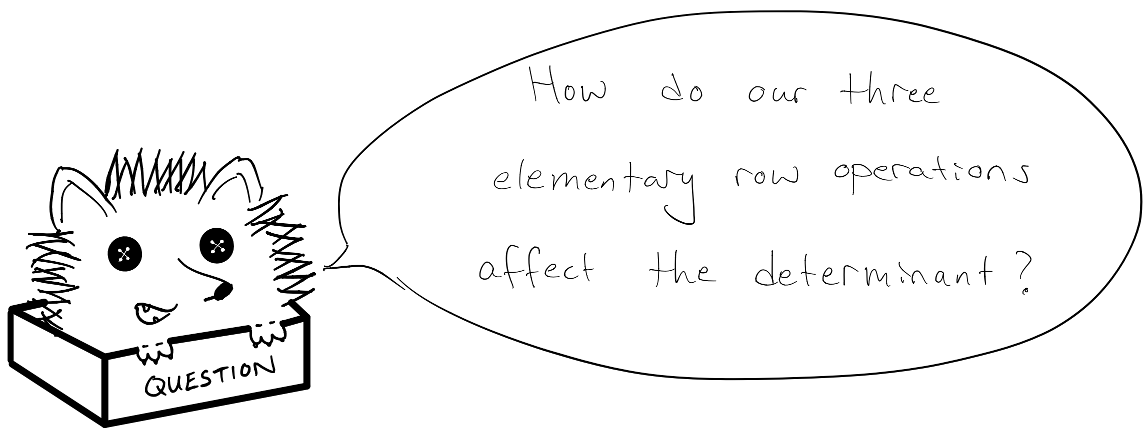
§ 5.2 - Determinants by Row Operations



Let's answer this question through some examples!

① Multiply a row by a scalar t.

Ex:

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \xrightarrow{R_1 \cdot 2} \begin{bmatrix} 6 & 14 \\ 2 & 5 \end{bmatrix}$$

Det = 1 $\qquad$ Det = 2

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \xrightarrow{R_2(-1)} \begin{bmatrix} 3 & 7 \\ -2 & -5 \end{bmatrix}$$

Det = -1

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \xrightarrow{R_1 \cdot t} \begin{bmatrix} 3t & 7t \\ 2 & 5 \end{bmatrix}$$

Det = t

$\boxed{\text{So it looks like multiplying a row by } t \text{ multiplies } \det(A) \text{ by } t \text{ as well!}}$

② Swapping two rows

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \xrightarrow{R_1 \updownarrow R_2} \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

Det = 1 　　　　Det = -1.

$$B = \begin{bmatrix} -1 & 4 \\ 5 & 1 \end{bmatrix} \xrightarrow{R_1 \updownarrow R_2} \begin{bmatrix} 5 & 1 \\ -1 & 4 \end{bmatrix}$$

Det = -21 　　　　Det = 21

$\boxed{\text{So it looks like swapping two rows multiplies } \det(A) \text{ by } -1}$

③ Adding a multiple of one row to another row.

Ex: $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 7 & 17 \\ 2 & 5 \end{bmatrix}$

Det = 1 Det = 1

$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 3 & 7 \\ -7 & -16 \end{bmatrix}$

Det = 1

$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \xrightarrow{R_1 + 201 R_2} \begin{bmatrix} 405 & 1012 \\ 2 & 5 \end{bmatrix}$

Det = 1

So it looks like adding / subtracting a multiple of one row from another doesn't affect det(A)!

These facts are extremely useful for computing determinants! Why? Let's see!

Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 6 & 5 \end{bmatrix}$, compute $\det(A)$.

Solution: Okay, this looks bad...

BUT as long as we keep track of how $\det(A)$ is affected, we can apply EROs to simplify A!

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 6 & 5 \end{bmatrix} \underbrace{\underset{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}}{\sim}}_{\text{Det(A) doesn't change}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}$$

This new matrix has the same determinant as A, but it has one new feature: it's upper triangular!

Therefore its determinant is the product of its diagonal entries: $\det = (1)(1)(-4) = -4$.

But this means that $\boxed{\det(A) = -4}$ as well!!

This gives us a new strategy for finding det(A):

1. Put A into REF using EROs
2. REF(A) is upper-triangular, so its determinant is the product of its diagonal entries.
3. Work back through your EROs to find det(A).

Ex: If $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 4 \\ 5 & 14 & 20 \end{bmatrix}$, find det(A).

Solution:

det is multiplied by ½

det is unchanged.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 4 \\ 5 & 14 & 20 \end{bmatrix} \overset{R_1 \cdot \frac{1}{2}}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 4 \\ 5 & 14 & 20 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 5R_1 \end{array} \sim$$

det is multiplied by (-1).

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & 4 & 5 \end{bmatrix} \underset{\sim}{R_2 \updownarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & -5 \end{bmatrix} \text{(REF)}$$

The determinant of the REF is $(1)(4)(-5) = -20$

To undo our ERO changes, we need to multiply by (-1) and multiply by 2.

$$\therefore \det(A) = (-20)(-1)(2) = \boxed{40}$$

Ex: If $A = \begin{bmatrix} 1 & 4 & 1 & 3 \\ 3 & 12 & 6 & 8 \\ -1 & -7 & 1 & 1 \\ 2 & 8 & 2 & 7 \end{bmatrix}$ find $\det(A)$.

Solution:

No change $\qquad\qquad$ $\cdot(-1)$

$$A = \begin{bmatrix} 1 & 4 & 1 & 3 \\ 3 & 12 & 6 & 8 \\ -1 & -7 & 1 & 1 \\ 2 & 8 & 2 & 7 \end{bmatrix} \begin{matrix} \sim \\ R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & 0 & 3 & -1 \\ 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_2 \updownarrow R_3}$$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -3 & 2 & 4 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has determinant $(1)(-3)(3)(1) = -9$

So, $\det(A) = (-9)(-1) = \boxed{9}$

Properties of Determinants

Theorem: Let A, B be $n \times n$ matrices.

1. A is invertible if and only if $\det(A) \neq 0$.

2. $\det(A^T) = \det(A)$

3. If $t \in \mathbb{R}$, then $\det(tA) = t^n \det(A)$

4. $\det(AB) = \det(A)\det(B)$

5. If A is invertible, $\det(A^{-1}) = \dfrac{1}{\det(A)}$

We can actually use 4 to get a slick proof of 5:

If A is invertible, then $AA^{-1} = I$.

Thus, $\det(AA^{-1}) = \det(I) = 1$

But $\det(AA^{-1}) = \det(A)\det(A^{-1})$ by 4, so

$$\det(A)\det(A^{-1}) = 1 \implies \det(A^{-1}) = \frac{1}{\det(A)}$$

$\blacksquare$

Note: 4 is also extremely useful for showing that certain matrices are invertible.

Ex: If A, B are $n \times n$ matrices and AB is invertible, then both A and B are invertible.

Proof: If AB is invertible, then $\det(AB) \neq 0$.

Hmm... but $\det(AB) = \det(A)\det(B)$, so $\det(A)\det(B) \neq 0$. This means $\det(A) \neq 0$ and $\det(B) \neq 0$, so A and B are invertible! $\blacksquare$