Chapter 5 - Determinants

§5.1 - Determinants by Cofactors

In class we saw this cute formula for the inverse of a 2×2 matrix:

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

It really does work! (Don't believe me? Check!)

However, this formula doesn't make sense if $ad-bc=0$.

Thus, the number $ad-bc$ determines whether or not $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse

For this reason we call it the determinant of A.

Definition: The determinant of a 2×2 matrix

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $\boxed{\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}}$

We have that A is invertible if and only if $\det(A) \neq 0$.

Ex: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $\det(A) = (1)(4) - (3)(2)$

$= \underline{-2}$

Since $\det(A) \neq 0$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is <u>invertible</u>.

Ex: If $A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$, then $\det(A) = (4)(1) - (-2)(-2)$

$= \underline{0}$

Since $\det(A) = 0$, $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ is <u>NOT invertible</u>.

We can also discuss determinants of $n\times n$ matrices for $n \geq 2$. Let's first look at 3×3's

Definition: The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is $\det(A) = |A|$ given by

$$\boxed{a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

This definition looks a little weird, but let's disect it!

- The coefficients a_{11}, a_{12}, a_{13} are the entries in the first row of A.
- The sign on each coefficient alternates $(+,-,+)$
- Each coefficient is multiplied by the determinant of the matrix obtained by deleting the row and column containing that coefficient.

Ex: If $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, then

$$\det(A) = 1 \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$$

$$= 1(1\cdot 1 - 1\cdot 3) - 3(2\cdot 1 - 1\cdot 0) + 0(2\cdot 3 - 1\cdot 0)$$

$$= 1(1-3) - 3(2-0) + 0(6-0)$$

$$= 1(-2) - 3(2)$$

$$= \boxed{-8}$$

Just like for 2×2 matrices, a 3×3 matrix A is invertible if and only if $\det(A) \neq 0$.

Since $\det\left(\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}\right) = -8 \neq 0$, this matrix is invertible!

Ex: If $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}$, then

$$\det(A) = 1\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \underbrace{\cancel{0\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}}}_{=0}$$

$$= 1(2-0) - 2(2-0) + 0$$

$$= 2 - 4$$

$$= \boxed{-2}$$

Since $\det(A) = -2 \neq 0$, $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}$ is invertible!

This method can be extended to define the determinant for any $n \times n$ matrix!

Once again, we can use the coefficients in the first row, alternate their signs, and multiply by the determinant of the matrix obtained by deleting the coefficient's row and column.

Ex: If $A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix}$, then

$$\det(A) = 3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\cancel{+ 0 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} \cancel{- 0 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix}}$$

$$= 3 \left(1 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \cancel{- 0 \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}} \cancel{+ 0 \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix}} \right)$$

$$- 2 \left(1 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \cancel{- 0 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}} \cancel{+ 0 \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}} \right)$$

$$= 3\Big(1(2-0)\Big) - 2\Big(1(2-0)\Big)$$

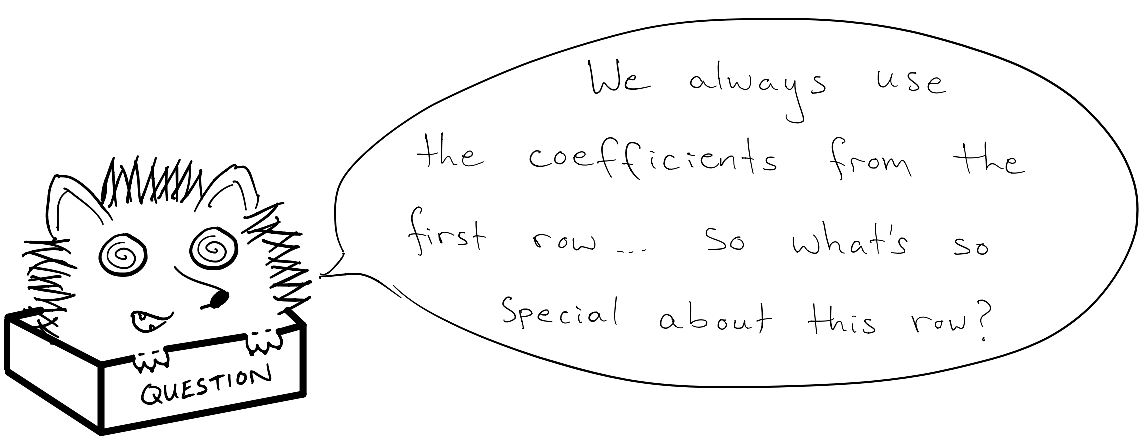
$$= 3(2) - 2(2)$$

$$= \boxed{2}$$

As in the 2×2 and 3×3 cases, an $n\times n$ matrix A is invertible if and only if $\det(A)\neq 0$.

Thus, we can add the statement "$\det(A)\neq 0$" to our BIG THEOREM from Chapter 3.

In particular, the 4×4 matrix in the above example is invertible, as its determinant is 2 $(\neq 0)$



Nothing is special about the first row! In fact, we can expand about ANY row or column as long as our signs are correct:

$\underline{3\times 3}$, $\underline{4\times 4}$, $\underline{5\times 5}$, etc...

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}, \quad \begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

Ex: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}$ as in one of our previous examples. Find $\det(A)$ using

(a) the second row

(b) the third column

Solution:

(a) $\det(A) = -1\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - \cancel{0\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}$

$= -1(4-0) + 1(2-0)$

$= \boxed{-2}$

(b) $\det(A) = \cancel{0\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}} - \cancel{0\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}} + 2\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$

$= 2(1-2)$

$= \boxed{-2}$

Good strategy: Pick a row or column with lots of zeros!

Ex: If $A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix}$, let's use the third column to find $\det(A)$!

$$\det(A) = \cancel{0\begin{vmatrix} ? \end{vmatrix}} - \cancel{0\begin{vmatrix} ? \end{vmatrix}} + 2\begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix} - \cancel{0\begin{vmatrix} ? \end{vmatrix}}$$

$$= 2\left(\cancel{0\begin{vmatrix} ? \end{vmatrix}} - \cancel{0\begin{vmatrix} ? \end{vmatrix}} + 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}\right)$$

$$= 2\left(1(3-2)\right)$$

$$= \boxed{2}$$

(MUCH FASTER!)

Applications:

(1) If A has a row or column of zeros, then $\det(A) = 0$. Why?

We can expand about that row or column to get every term $= 0$!

It then follows that if A has a row or column of zeros, then A is NOT invertible

(2) $\det(A) = \det(A^T)$. <u>Why</u>?

Since the rows of A are the columns of A^T, expanding A along the 1^{st} row is the same as expanding A^T along the 1^{st} column!

(3) If $A = (a_{ij})$ is upper-triangular,

(e.g., $\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$, etc...)

then $\boxed{\det(A) = a_{11} a_{22} \cdots a_{nn}}$, the product of the diagonal entries. <u>Why</u>?

You'll discuss this in Wednesday's tutorial.

(**Note:** The same result applies to lower-triangular matrices by taking a transpose.)