§3.5 - Inverse Matrices and Mappings

Consider the map $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that rotates vectors counterclockwise by θ radians.

We can "undo" this transformation by applying the map

$$R_{-\theta} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

(rotate vectors by θ clockwise). That is,

$$(R_\theta \circ R_{-\theta})(x_1, x_2) = (R_{-\theta} \circ R_\theta)(x_1, x_2) = (x_1, x_2).$$

for all vectors $\vec{x}$, so $\boxed{R_\theta \circ R_{-\theta} = R_{-\theta} \circ R_\theta = id}$.

Definition: A linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is **invertible** if there is a linear map $M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\boxed{M \circ L = L \circ M = id.}$$

We say that M is the **inverse** of L and write

$$\boxed{M = L^{-1}.}$$

Ex: From above, we have that R_θ is invertible and $R_\theta^{-1} = R_{-\theta}$.

Ex: If $\mathrm{Refl}_{\vec{n}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a reflection over a line, then $\mathrm{Refl}_{\vec{n}} \circ \mathrm{Refl}_{\vec{n}} = id$ (reflecting twice returns the original vector).

Thus, $\mathrm{Refl}_{\vec{n}}$ is invertible and $(\mathrm{Refl}_{\vec{n}})^{-1} = \mathrm{Refl}_{\vec{n}}$

Ex: On assignment 8, you showed that with

$$L(x_1, x_2) = (x_1 + x_2,\ x_1 + 2x_2),$$

$$M(x_1, x_2) = (2x_1 - x_2,\ x_2 - x_1)$$

We get $(L \circ M)(x_1, x_2) = (M \circ L)(x_1, x_2) = (x_1, x_2)$

$\therefore$ L is invertible and $L^{-1} = M$.

Also, M is invertible and $M^{-1} = L$

Remark: If $M = L^{-1}$, then $M \circ L = L \circ M = id$,

so
$$[M][L] = [M \circ L] = [id] = I$$
$$[L][M] = [L \circ M] = [id] = I$$

This allows us to talk about inverses in the matrix world!

Definition: If A is an $n \times n$ matrix, then A is invertible if there is an $n \times n$ matrix B such that

$$\boxed{AB = BA = I}$$

We say that B is the inverse of A, and write

$$\boxed{B = A^{-1}.}$$

Ex: If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, then

$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so A is invertible and $A^{-1} = B$

Ex: If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & -1 \end{bmatrix}$, then

$AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so A is invertible and $A^{-1} = B$.

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is NOT invertible. Why?

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ were its inverse, then

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ 0 & \boxed{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \boxed{1} \end{bmatrix}$$

Uh oh... these aren't equal.

$\therefore$ No inverse exists.

Properties: Let A, B be $n \times n$ invertible matrices.

(1) The inverse for A is unique.

(2) $(A^{-1})^{-1} = A$.

(3) $(AB)^{-1} = B^{-1}A^{-1}$.

(4) If $t \in \mathbb{R}$, $t \neq 0$, then $(tA)^{-1} = \frac{1}{t} A^{-1}$.

(5) $(A^T)^{-1} = (A^{-1})^T$

Proof:

(1) If C and D are both inverses for A, then

$$CA = AC = I \quad \text{and} \quad DA = AD = I$$

So $C = CI = C(AD) = (CA)D = ID = D$

$\therefore C = D!$

(2) Since $A^{-1}A = AA^{-1} = I$, we have that A is the inverse of A^{-1}, so $(A^{-1})^{-1} = A$.

(3) We must show that $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$

Indeed, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

(4), (5) Exercises ∎

Finding A^{-1}

Suppose we want to find the inverse X of a 2×2 matrix A.

Let $\vec{x}_1, \vec{x}_2$ be the columns of X: $X = [\vec{x}_1 \ \vec{x}_2]$

Then

$$AX = I \quad \Rightarrow \quad A[\vec{x}_1 \ \vec{x}_2] = I$$

$$\Rightarrow \quad [A\vec{x}_1 \ \ A\vec{x}_2] = [\vec{e}_1 \ \ \vec{e}_2]$$

$$\Rightarrow \quad A\vec{x}_1 = \vec{e}_1 \quad \& \quad A\vec{x}_2 = \vec{e}_2$$

$\therefore$ We need to solve $[A \mid \vec{e}_1]$, $[A \mid \vec{e}_2]$

We can solve both systems by putting $\vec{e}_1$ and $\vec{e}_2$ on the right hand side!

i.e., reduce $[A \mid \vec{e}_1 \ \vec{e}_2] = [A \mid I]$ to RREF

(see Q4 of A6).

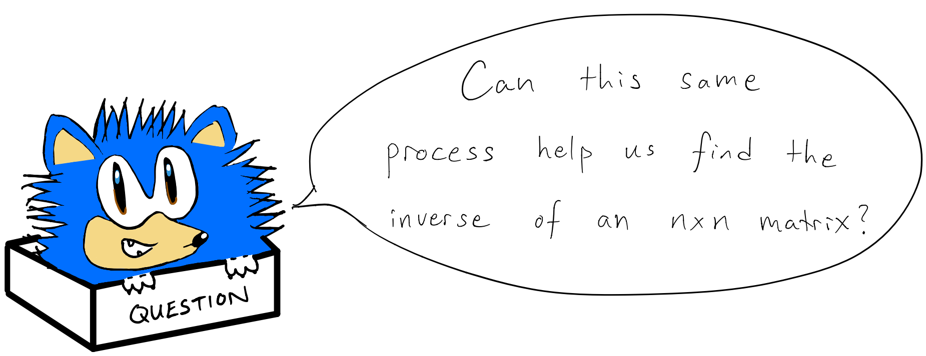
Ex: If $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$, then

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \underset{R_2 - R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 \end{array}\right] \underset{R_2 \cdot \frac{1}{3}}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/3 & 1/3 \end{array}\right]$$

These are the columns $\vec{x}_1$ and $\vec{x}_2$ of the inverse matrix!

$$\therefore \boxed{A^{-1} = \begin{bmatrix} 1 & 0 \\ -1/3 & 1/3 \end{bmatrix}}$$

Sure enough, $\begin{bmatrix} 1 & 0 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



Yes! There is nothing going on in the 2×2 setting that cannot be extended to $n \times n$ matrices.

The process: To find the inverse of an $n \times n$ matrix A ...

1. Put $[A \mid I_n]$ into RREF

2. If we get $[I_n \mid B]$, then A is invertible and $A^{-1} = B$.

3. If we don't get I_n on the left-hand side, then A is NOT invertible.

Ex: Find A^{-1} or explain why A is not invertible.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$

Solution:

(a) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \underset{R_2 - 2R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$ (RREF)

Since we don't have I_2 on the left, A is not invertible.

(b)
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \underset{R_3 - R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \underset{R_3(-1)}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \underset{\sim}{R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \underset{\sim}{R_1 - 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \text{(RREF)}$$

Since we get I_3 on the left, A is invertible.

Moreover, $\boxed{A^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}}.$

Note: by the above process, A is invertible if and only if $\text{RREF}(A) = I_n$.

BUT we know from Q5 of Assignment 6 that this is equivalent to the statements in our "BIG THEOREM". Thus, BIG THEOREM gets an upgrade!

BIG THEOREM (V2): If A is an $n \times n$ matrix, then the following are equivalent:

(1) $RREF(A) = I_n$

(2) A is invertible

(3) $Rank(A) = n$

(4) The columns of A are linearly independent

(5) $A\vec{x} = \vec{0}$ has a unique solution $(\vec{x} = \vec{0})$

(6) $Null(A) = \{\vec{0}\}$

(7) The columns of A span $\mathbb{R}^n$

(8) $[A \mid \vec{b}]$ is consistent for all $\vec{b} \in \mathbb{R}^n$.

(9) $Col(A) = \mathbb{R}^n$

(10) $Row(A) = \mathbb{R}^n$

(11) The columns of A form a basis for $\mathbb{R}^n$.

(12) L is invertible

(13) $Range(L) = \mathbb{R}^n$

(14) $Null(L) = \{\vec{0}\}$

Here, L denotes the linear map $L(\vec{x}) = A\vec{x}$