§3.4 – Special Subspaces of Systems and Mappings

The Key idea from Chapter 3: there is an intimate connection between

Linear function World	and	Matrix World.

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \longleftrightarrow \quad [L] \quad m \times n$$

$$L(\vec{x}) = [L]\vec{x}$$

How do our Chapter 2 notions in matrix world (homogeneous systems, rank, etc...) match with ideas in linear function world? We can better understand such connections by studying subspaces associated to L and its corresponding matrix.

Throughout, $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ will be a linear map and A will be its $m \times n$ matrix $[L]$.

Recall: A subspace of $\mathbb{R}^n$ is a non-empty subset of $\mathbb{R}^n$ that is

(1) closed under addition, and

(2) closed under scalar multiplication.

Nullspace

If A is an $m \times n$ matrix, then the set

$$\boxed{\mathrm{Null}(A) = \left\{ \vec{x} \in \mathbb{R}^n \,\middle|\, A\vec{x} = \vec{0} \right\}}$$

of all solutions to the homogeneous system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

This is known as the nullspace of A.

Ex: What is the nullspace of $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 5 \end{bmatrix}$?

Solution: It's the set of all solutions to $A\vec{x} = \vec{0}$ (i.e., solutions to $[A \mid \vec{0}]$) which we can obtain by finding RREF(A).

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 5 \end{bmatrix} \underset{R_3 - R_1}{\sim} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \sim \\ R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

If $x_2 = s$ and $x_4 = t$ $(s, t \in \mathbb{R})$, then the solutions are

$$\vec{x} = \begin{bmatrix} -s-t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

$$\therefore \boxed{\operatorname{Null}(A) = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}}$$

<u>What is the analogue of nullspace in linear function world??</u>

Well, if $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map with matrix A, then $A\vec{x} = \vec{0}$ if and only if $L(\vec{x}) = \vec{0}$

So Null(A) is the same as

$$\boxed{\text{Null}(L) = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; L(\vec{x}) = \vec{0} \right\}}$$

which we will call the nullspace of L.

Ex: What is the nullspace of the linear map

$$L(x_1, x_2, x_3) = (x_1 + x_2,\ x_2 + x_3)\ ?$$

Solution: The matrix for L is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

The nullspace of L is the same as Null(A), which we can get by solving $A\vec{x} = \vec{0}$.

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \implies \text{Solution is } \vec{x} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\ t \in \mathbb{R}$$

$$\therefore \boxed{\text{Null}(L) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}}$$

Column Space of A and Range of L

Recall that the codomain of a linear transformation $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is $\mathbb{R}^m$, the set that L maps into.

$$\boxed{\text{But does } L \text{ actually hit every vector in its codomain?}}$$

Ex: If $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, then the linear map $\text{Proj}_{\vec{v}}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ sends each vector $\vec{x} \in \mathbb{R}^3$ to $\text{Proj}_{\vec{v}} \vec{x}$, which is always a multiple of $\vec{v}$.

So $\text{Proj}_{\vec{v}}$ hits all multiples of $\vec{v}$, but nothing else (e.g., $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is not hit by $\text{Proj}_{\vec{v}}$)

Ex: Thinking geometrically, its easy to see that rotations, reflections, and stretches/compressions on $\mathbb{R}^2$ do hit every vector in their codomain!

Definition: The range of a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is

$$\boxed{\text{Range}(L) = \{ L(\vec{x}) : \vec{x} \in \mathbb{R}^n \}.}$$

This is the set of vectors in $\mathbb{R}^m$ that L actually hits.

Ex: By the above examples, we have

(i) $\text{Range}(\text{Proj}_{\vec{v}}) = \text{Span}\{\vec{v}\}$

(ii) $\text{Range}(R_\theta) = \mathbb{R}^2$

(iii) $\text{Range}(\text{Refl}_{\vec{n}}) = \mathbb{R}^2$

What is the analogue of range in the matrix world??

Since $L(\vec{x}) = A\vec{x}$, we have that

$$\text{Range}(L) = \{ L(\vec{x}) : \vec{x} \in \mathbb{R}^n \}$$

$$= \{ \underbrace{A\vec{x}}_{} : \vec{x} \in \mathbb{R}^n \}$$

= linear combination of columns of A.

$$= \left\{ \begin{array}{l} \text{all linear combinations} \\ \text{of the columns of } A \end{array} \right\} = \operatorname{Span}\left\{ \text{columns of } A \right\}$$

Let's give this last set a proper name!

Definition: The column space of an $m \times n$ matrix A is

$$\boxed{\operatorname{Col}(A) = \{ A\vec{x} : \vec{x} \in \mathbb{R}^n \} = \operatorname{Span}\{ \text{columns of } A \}}$$

$\therefore$ If L is a linear map with matrix A, then

$$\boxed{\operatorname{Range}(L) = \operatorname{Col}(A).}$$

Ex: What is the range of the linear map

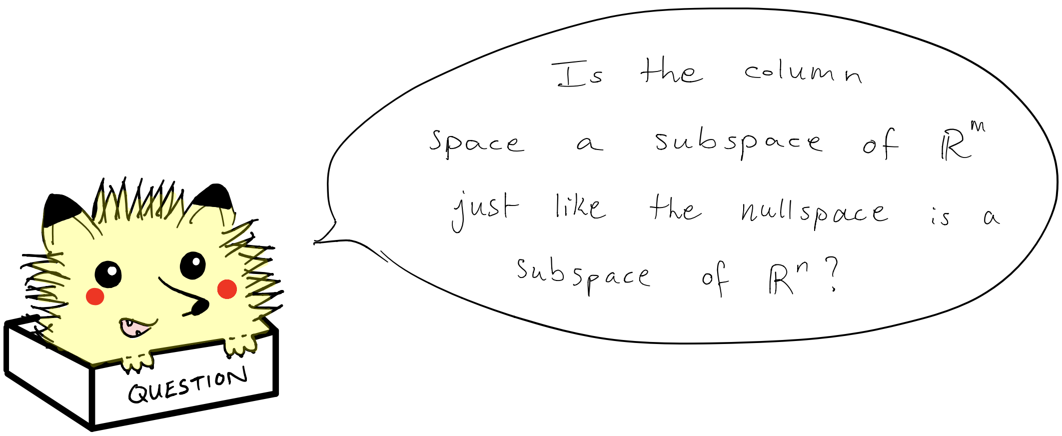
$$L(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)\ ?$$

Solution: From a previous example, the matrix for L is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Thus,

$$\operatorname{Range}(L) = \operatorname{Col}(A) = \boxed{\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$$

Ex: If $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 5 \end{bmatrix}$, then

$$\boxed{\operatorname{Col}(A) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \right\}}$$



Yes! In §1.7 we saw that $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ is always a subspace! So in particular,

$\operatorname{Col}(A) = \operatorname{Span}\{\text{columns of } A\}$ is a subspace.

Note: From Chapter 2 we know that a system $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ belongs to $\operatorname{Span}\{\text{columns of } A\}$.

In our new language:

$$\boxed{\begin{gathered} A\vec{x} = \vec{b} \text{ is consistent if and only if } \vec{b} \in \mathrm{Col}(A) \\ \left[\text{or equivalently, } \vec{b} \in \mathrm{Range}(L)\right] \end{gathered}}$$

Rowspace

In some cases we may be interested in knowing the span of the rows of a matrix.

Definition: If A is an $m \times n$ matrix, then the rowspace of A, $\mathrm{Row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the rows of A when regarded as column vectors.

Ex: If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, then

$$\mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Formally, $\boxed{\text{Row}(A) = \text{Col}(A^T) = \{A^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\}}$

Magic Fact about Row Spaces

EROs don't change the row space!

In particular, $\text{Row}(A) = \text{Row}(\text{RREF}(A))$

Ex: If $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 5 \end{bmatrix}$, then

$$\text{RREF}(A) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so}$$

$$\text{Row}(A) = \text{Row}(\text{RREF}(A)) = \boxed{\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}}$$

This is clearly NOT a basis for Row(A).

Is there some way we can obtain a basis directly?

Subspace	Subspace of	How to get a Basis
$\text{Col}(A) = \text{Range}(L)$	$\mathbb{R}^m$ (codomain)	Use the columns of A that have leading 1's in RREF(A)
$\text{Row}(A) = \text{Col}(A^T)$	$\mathbb{R}^n$ (domain)	Use the non-zero rows of RREF(A)
$\text{Null}(A) = \text{Null}(L)$	$\mathbb{R}^n$ (domain)	Solve $A\vec{x} = \vec{0}$ and use the vectors on each parameter.

Ex: If $A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 9 & 5 \\ 3 & 13 & 6 \end{bmatrix}$ and $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$,

find bases for Col(A), Row(A), and Null(A).

Solution: Since Columns 1 and 2 have pivots in RREF(A),

$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 13 \end{bmatrix} \right\}}$ is a basis for Col(A).

Since rows 1 and 2 of RREF(A) are non-zero,

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}}$$ is a basis for Row(A).

The solution to $A\vec{x} = \vec{0}$ is $\vec{x} = t\begin{bmatrix} 11 \\ -3 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$, so

$$\boxed{\left\{ \begin{bmatrix} 11 \\ -3 \\ 1 \end{bmatrix} \right\}}$$ is a basis for Null(A).

Ex: If $A = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 1 & 8 & 10 \\ 2 & 4 & 1 & 11 & 14 \end{bmatrix}$ and

$\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, find bases for

Col(A), Row(A), and Null(A).

Solution: A basis for Col(A) is $\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}}$

A basis for Row(A) is $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \\ 6 \end{bmatrix} \right\}}$

The general solution for $A\vec{x} = \vec{0}$ is

$$\vec{x} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \quad r, s, t \in \mathbb{R},$$

So a basis for $\mathrm{Null}(A)$ is $\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix} \right\}}$

Note: If there are no parameters in the solution to $A\vec{x} = \vec{0}$, then $\mathrm{Null}(A) = \{\vec{0}\}$. In this case we say that the nullspace of A is trivial.

Observations:

(1) $\mathrm{Col}(A)$ contains a basis vector for every pivot in $\mathrm{RREF}(A)$. Thus,

$$\boxed{\dim \mathrm{Col}(A) = \mathrm{Rank}(A).}$$

(2) $\mathrm{Row}(A)$ contains a basis vector for every pivot in $\mathrm{RREF}(A)$. Thus,

$$\boxed{\dim \mathrm{Row}(A) = \mathrm{Rank}(A).}$$

(3) Null(A) contains a basis vector for every parameter in the solution to $A\vec{x} = \vec{0}$. Since there are $n - \text{Rank}(A)$ parameters,

$$\boxed{\dim \text{Null}(A) = n - \text{Rank}(A)}$$

\# of columns.

This leads us to the following FUNDAMENTAL result:

<u>Theorem</u> [Rank-Nullity Theorem]:

If A is an $m \times n$ matrix, then

$$\text{Rank}(A) + \dim \text{Null}(A) = n.$$

<u>Ex</u>: A 5×4 matrix A has rank 3, what is the dimension of its nullspace?

<u>Solution</u>: Since $\text{Rank}(A) + \dim \text{Null}(A) = 4$, we have that

$$\dim \text{Null}(A) = 4 - \text{Rank}(A)$$
$$= 4 - 3 = \boxed{1}.$$

<u>Ex</u>: A 10×17 matrix A has a 14-dimensional nullspace. What is $\dim \text{Col}(A)$?

Solution: Note that since $\text{Rank}(A) + \dim \text{Null}(A) = 17$,

we have
$$\begin{aligned}\text{Rank}(A) &= 17 - \dim \text{Null}(A)\\ &= 17 - 14\\ &= 3\end{aligned}$$

$$\therefore \dim \text{Col}(A) = \text{Rank}(A) = \boxed{3}.$$

Ex: A 6×6 matrix A has 1-dimensional nullspace. Are the columns of A linearly independent?

Solution: By the Rank-Nullity Theorem,

$$\text{Rank}(A) = 6 - \dim \text{Null}(A) = 6 - 1 = 5$$

Thus, $\text{Rank}(A) < \#$ of columns $= 6$, so by the results of Chapter 2, the columns are linearly dependent.

[In fact this argument shows that the columns of A are linearly independent if and only if $\dim \text{Null}(A) = 0$ (i.e., $\text{Null}(A) = \{\vec{0}\}$). This makes sense: if there were some non-zero $\vec{x} \in \text{Null}(A)$, then we would have $A\vec{x} = \vec{0}$, meaning there is a non-trivial linear combination of A's columns that equals $\vec{0}$!]

Cute Application: If A is an $m \times n$ matrix, then $\text{Rank}(A) = \text{Rank}(A^T)$.

Why? Because $\text{Rank}(A) = \dim \text{Row}(A)$

and $\text{Rank}(A^T) = \dim \text{Col}(A^T)$.

Since $\text{Row}(A) = \text{Col}(A^T)$, we get

$$\text{Rank}(A) = \dim \text{Row}(A) = \dim \text{Col}(A^T) = \text{Rank}(A^T).$$

$\blacksquare$