§3.3 - Geometrical Transformations

Many linear transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$ are easy to visualize, and play an important role in modelling and structural analysis.

Here we record several such transformations and find their matrices. The name of the game? Determine what happens to the standard basis vectors!

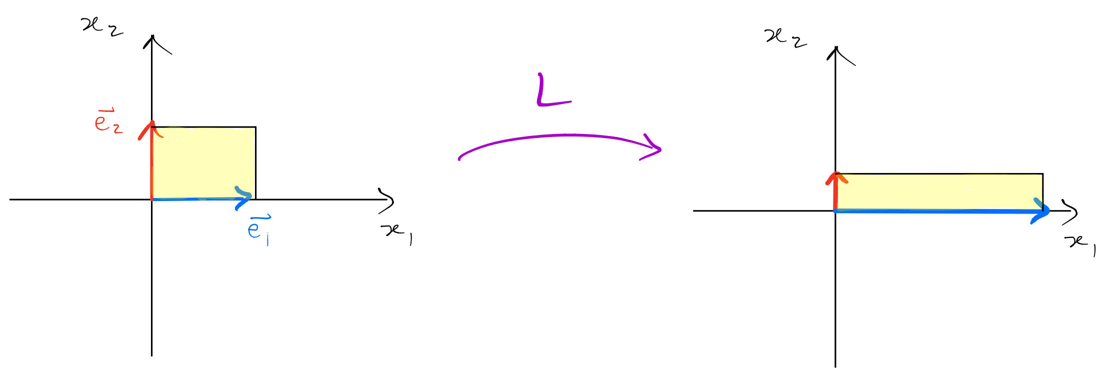
Stretches

Let s, t be > 0 and consider the linear transformation L that stretches/contracts vectors $\vec{x} \in \mathbb{R}^2$ by s in the x_1-direction and t in the x_2-direction.

What is $[L]$?

Well ... $\begin{cases} L(\vec{e}_1) = s\vec{e}_1 = \begin{bmatrix} s \\ 0 \end{bmatrix} \\ L(\vec{e}_2) = t\vec{e}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix} \end{cases} \Longrightarrow \boxed{[L] = \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}}$.

Ex: If $L:\mathbb{R}^2 \to \mathbb{R}^2$ is a stretch by 2 in the x_1-direction and a contraction by $\frac{1}{2}$ in the x_2-direction, then $[L] = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$



If L stretches/contracts the x_1- and x_2-directions by the same factor t (so $[L] = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$), we simply call L a stretch/contraction by a factor of t.

In $\mathbb{R}^3$: if L stretches or contracts the x_1-, x_2-, and x_3-directions by r, s, t, respectively, then

$$\boxed{[L] = \begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{bmatrix}.}$$

One particular example: a stretch by a factor of 1 leaves every vector $\vec{x}$ unchanged. This transformation $\mathrm{id} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$\mathrm{id}(\vec{x}) = \vec{x},$$

and for this reason we call it the identity map.

Since $\mathrm{id}(\vec{e_i}) = \vec{e_i}$, its matrix is

$$\boxed{[\mathrm{id}] = [\vec{e_1}\ \vec{e_2}\ \cdots\ \vec{e_n}] = I_n}$$

Projections / Perpendiculars

In §1.4 we saw that if $\vec{x} \neq \vec{0}$, then

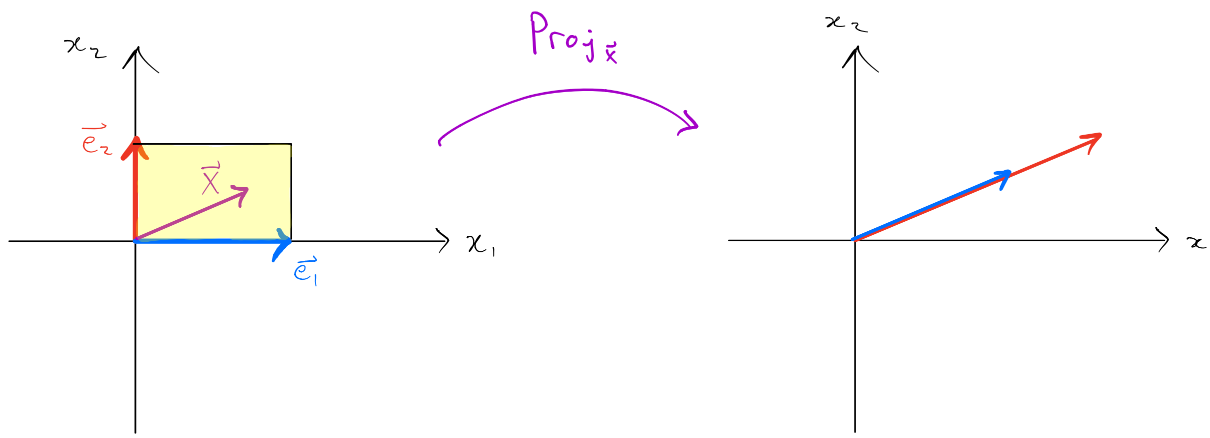
$$\boxed{\begin{aligned} \mathrm{Proj}_{\vec{x}}(\vec{y} + \vec{z}) &= \mathrm{Proj}_{\vec{x}}\vec{y} + \mathrm{Proj}_{\vec{x}}\vec{z} \\ \mathrm{Proj}_{\vec{x}}(t\vec{y}) &= t\,\mathrm{Proj}_{\vec{x}}\vec{y} \end{aligned}}$$

for all $\vec{y}, \vec{z} \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.

Thus, the map $\mathrm{Proj}_{\vec{x}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ that sends $\vec{y}$ to $\mathrm{Proj}_{\vec{x}}\vec{y}$ is linear.

Its matrix $[\text{Proj}_{\vec{x}}]$ can be found by calculating

$\text{Proj}_{\vec{x}}\,\vec{e}_1$, $\text{Proj}_{\vec{x}}\,\vec{e}_2$, ..., $\text{Proj}_{\vec{x}}\,\vec{e}_n$ (see Q2 of Assignment 8)



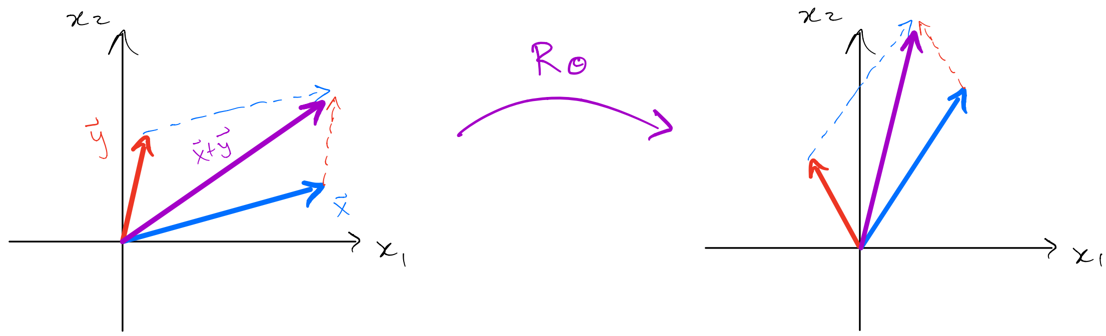
Rotations

Let $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the transformation that rotates all vectors counterclockwise about the origin by θ radians.

Note: It doesn't matter if a vector is stretched and then rotated, or rotated and then stretched.

(i.e., $R_\theta(t\vec{x}) = tR_\theta(\vec{x})$, so R_θ satisfies (L2))

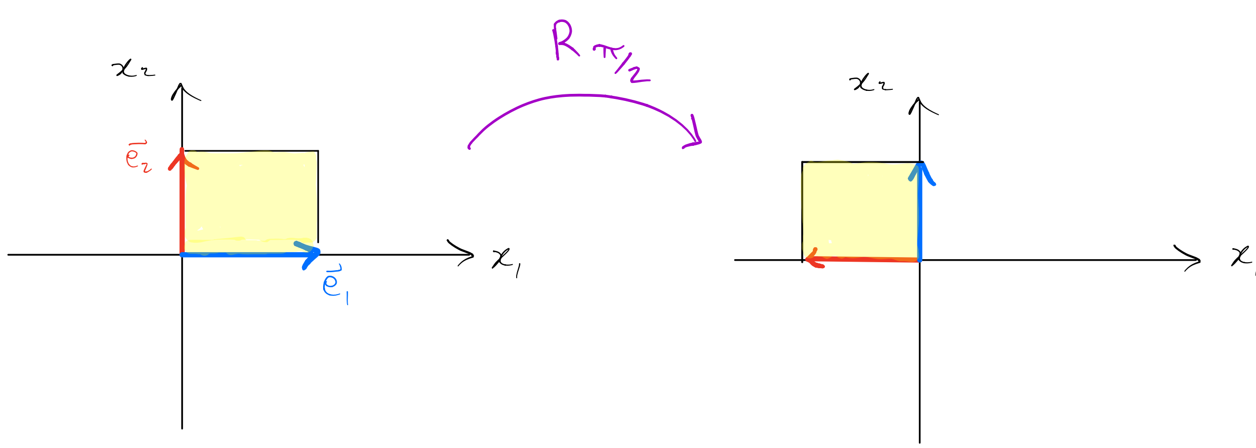
Furthermore, if $\vec{x}$ and $\vec{y}$ are vectors, then $\vec{x}+\vec{y}$ rotates with $\vec{x}$ and $\vec{y}$ as we apply R_θ



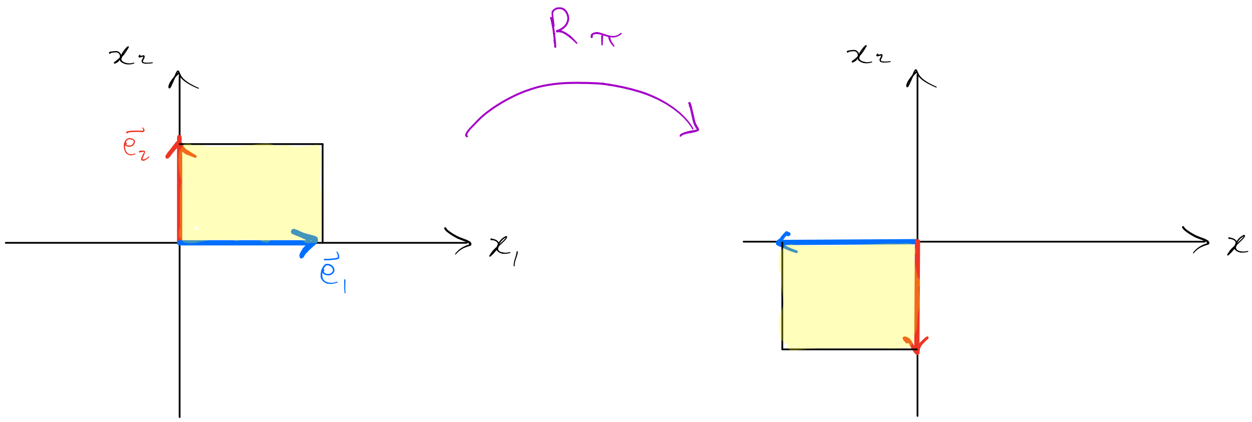
So it doesn't matter if we first add and then rotate, or rotate and then add (i.e., $R_\theta(\vec{x}+\vec{y}) = R_\theta(\vec{x}) + R_\theta(\vec{y})$)

$\therefore$ (L1) holds, so <u>R_θ is linear!</u>

What is $[R_\theta]$? Answering this question is easy for some angles:



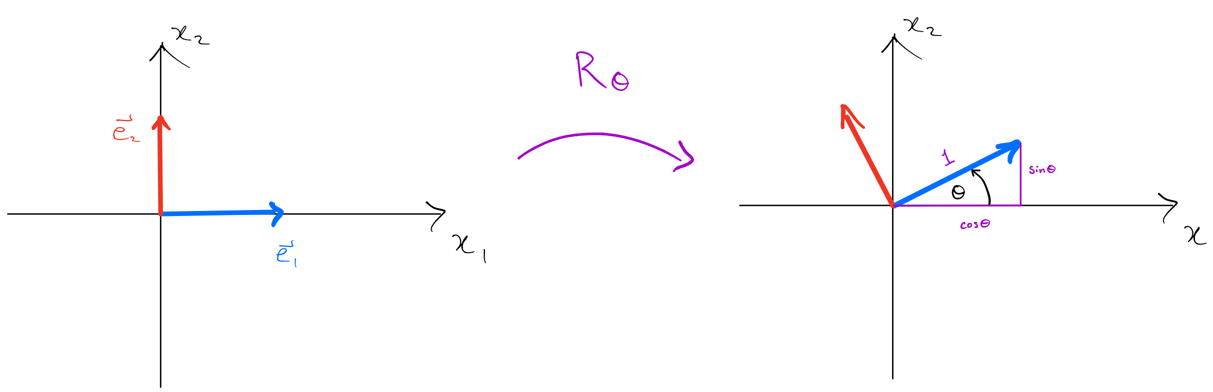
$$R_{\pi/2}(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad R_{\pi/2}(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \implies \underline{[R_{\pi/2}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$



$$R_{\pi}(\vec{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad R_{\pi}(\vec{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \implies \underline{[R_{\pi}] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

What about for a general θ?

From our knowledge of the unit circle, we know that $R_\theta(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



What's $R_\theta(\vec{e}_2)$? It should be the same as

$$R_{\theta+\pi/2}(\vec{e}_1) = \begin{bmatrix} \cos(\theta+\pi/2) \\ \sin(\theta+\pi/2) \end{bmatrix}$$

(Note: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\sin(\alpha+\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$)

$$= \begin{bmatrix} \cos\theta\cos(\pi/2) - \sin\theta\sin(\pi/2) \\ \sin\theta\cos(\pi/2) + \cos\theta\sin(\pi/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cdot 0 - \sin\theta\cdot 1 \\ \sin\theta\cdot 0 + \cos\theta\cdot 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Thus, $\boxed{[R_\theta] = [R_\theta(\vec{e}_1)\ \ R_\theta(\vec{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$

Ex: Determine $[R_{2\pi/3}]$ and use it to find the vector obtained by rotating $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ by $\frac{2\pi}{3}$ radians counterclockwise.

Solution $[R_{\frac{2\pi}{3}}] = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix}$

$$= \boxed{\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}}$$

We have $R_{\frac{2\pi}{3}}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = [R_{\frac{2\pi}{3}}]\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

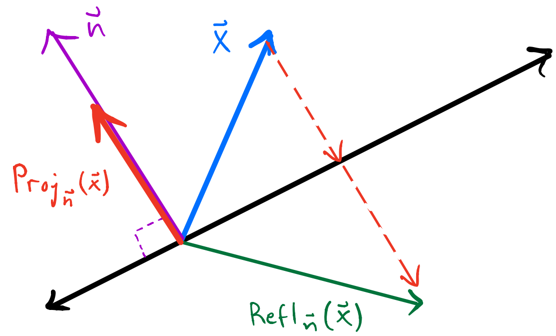
$$= \boxed{\begin{bmatrix} -1-2\sqrt{3} \\ \sqrt{3}-2 \end{bmatrix}}$$

Reflection Over a Line in $\mathbb{R}^2$

Let's say we have a line in $\mathbb{R}^2$ that passes through the origin, and suppose $\vec{n}$ is a normal vector for the line (so $\vec{n} \cdot \underbrace{\vec{d}}_{\text{direction vector}} = 0$)

If $\mathrm{Refl}_{\vec{n}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denotes the map that reflects each vector $\vec{x}$ over this line, then

$$\boxed{\mathrm{Refl}_{\vec{n}}(\vec{x}) = \vec{x} - 2\,\mathrm{Proj}_{\vec{n}}(\vec{x})}$$



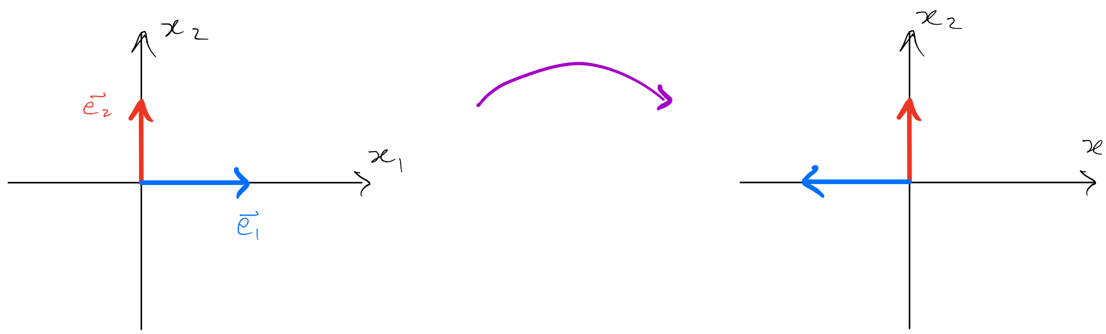
Since
$$\begin{aligned}\mathrm{Refl}_{\vec{n}}(\vec{x}) &= \vec{x} - 2\,\mathrm{Proj}_{\vec{n}}(\vec{x}) \\ &= \underbrace{\left(\mathrm{id} - 2\,\mathrm{Proj}_{\vec{n}}\right)}_{\text{Since id and } \mathrm{Proj}_{\vec{n}} \text{ are linear, so is this!}}(\vec{x})\end{aligned}$$

we have that $\mathrm{Refl}_{\vec{n}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is linear, and

$$\boxed{\left[\mathrm{Refl}_{\vec{d}}\right] = \left[\mathrm{id} - 2\,\mathrm{Proj}_{\vec{n}}\right] = I - 2\left[\mathrm{Proj}_{\vec{n}}\right]}$$

Ex: Find the matrix that represents a reflection across the x_2-axis in $\mathbb{R}^2$.

Solution: We can see right away that this map sends $\vec{e}_1$ to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, and $\vec{e}_2$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



So we get $\boxed{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$.

Let's see if our above approach gives the same result!

Since $\vec{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\perp$ to this line, we're looking for

$$[\mathrm{Refl}_{\vec{n}}] = I - 2\,[\mathrm{Proj}_{\vec{n}}]$$

We have

$$\mathrm{Proj}_{\vec{n}}(\vec{e}_1) = \frac{1}{1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathrm{Proj}_{\vec{n}}(\vec{e}_2) = \frac{0}{1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This means $[\mathrm{Proj}_{\vec{n}}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and hence

$$[\mathrm{Refl}_{\vec{n}}] = I - 2\,[\mathrm{Proj}_{\vec{n}}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

(Yay! It's the same!)

Ex: Find the matrix that represents a reflection across the line $\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix}$, $t \in \mathbb{R}$

Use this matrix to find the vector obtained by reflecting $\begin{bmatrix}2\\1\end{bmatrix}$ in this line.

Solution: $\vec{n} = \begin{bmatrix}1\\-1\end{bmatrix}$ is a normal vector.

We have $\quad \mathrm{Proj}_{\vec{n}}\, \vec{e}_1 = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\end{bmatrix}$

$$\mathrm{Proj}_{\vec{n}}\, \vec{e}_2 = \frac{-1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\end{bmatrix}$$

So $\left[\mathrm{Proj}_{\vec{n}}\right] = \begin{bmatrix}1/2 & -1/2\\-1/2 & 1/2\end{bmatrix}$, and hence

$$\left[\mathrm{Refl}_{\vec{n}}\right] = I - 2\left[\mathrm{Proj}_{\vec{n}}\right] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - 2\begin{bmatrix}1/2 & -1/2\\-1/2 & 1/2\end{bmatrix} = \boxed{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}}$$

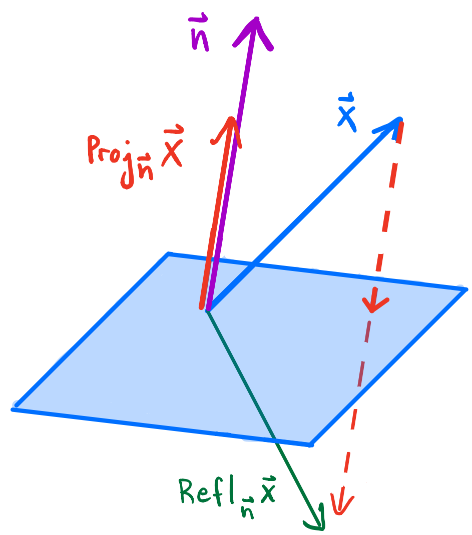
Thus, $\mathrm{Refl}_{\vec{n}}\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \left[\mathrm{Refl}_{\vec{n}}\right]\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \boxed{\begin{bmatrix}1\\2\end{bmatrix}}$

Reflection Over a Plane in $\mathbb{R}^3$

Let's say we have a plane $\vec{n}\cdot\vec{x} = 0$ in $\mathbb{R}^3$ that passes through the origin and has normal vector $\vec{n}$.

If now $\text{Refl}_{\vec{n}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ denotes the map that reflects every vector $\vec{x}$ over this plane, then

$$\boxed{\text{Refl}_{\vec{n}}(\vec{x}) = \vec{x} - 2\,\text{Proj}_{\vec{n}}(\vec{x})}$$



This means once again,

$$\boxed{[\text{Refl}_{\vec{n}}] = I - 2[\text{Proj}_{\vec{n}}]}$$

Ex: Consider the plane $x_1 + 2x_2 + x_3 = 0$ in $\mathbb{R}^3$. Find the matrix representing a reflection in this plane; then find the image of $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ under this map.

Solution: The normal vector is $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and we must find $[\mathrm{Refl}_{\vec{n}}] = I - 2[\mathrm{Proj}_{\vec{n}}]$.

Since

$$\mathrm{Proj}_{\vec{n}}\,\vec{e}_1 = \frac{1}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/6 \end{bmatrix}$$

$$\mathrm{Proj}_{\vec{n}}\,\vec{e}_2 = \frac{2}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$\mathrm{Proj}_{\vec{n}}\,\vec{e}_3 = \frac{1}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/6 \end{bmatrix}$$

We have $[\mathrm{Proj}_{\vec{n}}] = \begin{bmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & 2/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{bmatrix}$, and hence

$$[\mathrm{Refl}_{\vec{n}}] = I - 2[\mathrm{Proj}_{\vec{n}}]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & 2/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{bmatrix} = \boxed{\begin{bmatrix} 2/3 & -2/3 & -1/3 \\ -2/3 & -1/3 & -2/3 \\ -1/3 & -2/3 & 2/3 \end{bmatrix}}$$

The reflection of $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ in this plane is

$$\mathrm{Refl}_{\vec{n}}\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right) = [\mathrm{Refl}_{\vec{n}}]\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 & -1/3 \\ -2/3 & -1/3 & -2/3 \\ -1/3 & -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}}$$