§3.2 – Matrix Mappings and Linear Mappings

In high school you studied functions from $\mathbb{R}$ to $\mathbb{R}$.

Ex: The function $f(x) = x^2$ assigns to each x in the domain ($\mathbb{R}$) a unique value y in the codomain (also $\mathbb{R}$). We say that f maps x to y, or that y is the image of x under f.

In this section we will begin to study functions on higher dimensional spaces:

$$f : \underbrace{\mathbb{R}^n}_{\text{(domain)}} \longrightarrow \underbrace{\mathbb{R}^m}_{\text{(codomain)}}$$

These are sometimes called mappings, transformations, or operators.

Matrix Mappings

Let A be an $m \times n$ matrix. We know that if $\vec{x}$ is a vector in $\mathbb{R}^n$, then $A\vec{x}$ is a vector in $\mathbb{R}^m$.

This gives us a mapping $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$f_A(\vec{x}) = A\vec{x}.$$

We commonly call this the matrix mapping corresponding to A.

Note: If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a mapping that sends $\vec{x} \in \mathbb{R}^n$ to $\vec{y} \in \mathbb{R}^m$, it would be correct to write

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}.$$

Instead, however, we typically write $f(x_1, x_2, \ldots, x_n) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ or $f(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$. This looks a little cleaner!

Ex: If $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \\ 3 & 3 \end{bmatrix}$, then A is 3×2.

By above, we can view this as a matrix mapping

$$f_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

that sends $\vec{x}$ to $A\vec{x}$.

We have $f_A(2,1) = A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 9 \end{bmatrix}$

and $f_A(1,-1) = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$

Ex: If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$ then A is 2×3

This gives us a matrix mapping $f_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

We have ... $f_A(1,0,0) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$f_A(0,1,0) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Note: This makes sense! We know from §3.1 that

$A\vec{e_i} = i^{th}$ column of A.

$$f_A(1,1,0) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$f_A(2,0,0) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Notice that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{f_A(1,1,0) = f_A(1,0,0) + f_A(0,1,0)};$$

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{f_A(2,0,0) = 2 f_A(1,0,0)}.$$

$$\boxed{\text{This is no coincidence!}}$$

Theorem: If A is an $m \times n$ matrix, then the matrix mapping $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies

(L1): $f_A(\vec{x} + \vec{y}) = f_A(\vec{x}) + f_A(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$

(L2): $f_A(t\vec{x}) = t f_A(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$.

Proof: (L1): $f_A(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = f_A(\vec{x}) + f_A(\vec{y})$

(L2): $f_A(t\vec{x}) = A(t\vec{x}) = t(A\vec{x}) = t f_A(\vec{x})$ ∎

These two properties are <u>SUPER</u> important!

Let's see why!

Ex: Suppose A is a 3×3 matrix and we know

$$f_A(1,0,0) = (3,3,1),$$
$$f_A(0,1,0) = (-1,0,1)$$
$$f_A(0,0,1) = (0,2,0).$$

What is $f_A(1,4,2)$?

Solution: Since $\begin{bmatrix}1\\4\\2\end{bmatrix} = (1)\begin{bmatrix}1\\0\\0\end{bmatrix} + (4)\begin{bmatrix}0\\1\\0\end{bmatrix} + (2)\begin{bmatrix}0\\0\\1\end{bmatrix}$, we have

$$f_A(1,4,2) = f_A(1,0,0) + 4 f_A(0,1,0) + 2 f_A(0,0,1)$$

$$= \begin{bmatrix}3\\3\\1\end{bmatrix} + 4\begin{bmatrix}-1\\0\\1\end{bmatrix} + 2\begin{bmatrix}0\\2\\0\end{bmatrix} = \boxed{\begin{bmatrix}-1\\7\\5\end{bmatrix}}$$

<u>So why are (L1) and (L2) important?</u>

Because if a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies (L1) and (L2) and we know $f(\vec{x}_1)$, $f(\vec{x}_2)$, ..., $f(\vec{x}_k)$, then we can find

$$\boxed{f(t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k) = t_1 f(\vec{x}_1) + t_2 f(\vec{x}_2) + \cdots + t_k f(\vec{x}_k).}$$

For a matrix mapping we can say even more!

Since $f_A(\vec{e}_i) = A\vec{e}_i = i^{th}$ column of A, we have

$$\boxed{A = \begin{bmatrix} f_A(\vec{e}_1) & f_A(\vec{e}_2) & \cdots & f_A(\vec{e}_n) \end{bmatrix}}$$

So f_A is entirely determined by what it does to the standard basis!

Ex: From the previous example,

$f_A(1,0,0) = (3,3,1)$, $f_A(0,1,0) = (-1,0,1)$, $f_A(0,0,1) = (0,2,0)$.

So $A = \begin{bmatrix} f_A(\vec{e}_1) & f_A(\vec{e}_2) & f_A(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

Linear Mappings

We saw above that matrix mappings satisfy (L1) and (L2). Are there other maps with these properties?

Definition: If a function $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies

(L1) $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, and

(L2) $L(t\vec{x}) = tL(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$,

We call L a linear mapping or a linear transformation.

Ex: Show that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$f(x_1, x_2) = (x_2, x_1 - 2x_2)$ is linear.

Solution: We must show

(L1) $\underline{f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})}$ and (L2) $\underline{f(t\vec{x}) = t f(\vec{x})}$

for all $\vec{x}, \vec{y} \in \mathbb{R}^2$ and $t \in \mathbb{R}$.

(L1)
$$
\begin{aligned}
f(\vec{x}+\vec{y}) &= f(x_1+y_1, x_2+y_2) \\
&= \begin{bmatrix} x_2+y_2 \\ (x_1+y_1) - 2(x_2+y_2) \end{bmatrix} \\
&= \begin{bmatrix} x_2+y_2 \\ (x_1 - 2x_2) + (y_1 - 2y_2) \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} x_2 \\ x_1 - 2x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_1 - 2y_2 \end{bmatrix} = f(x_1, x_2) + f(y_1, y_2).$$

$\therefore$ (L1) holds.

(L2):
$$f(t\vec{x}) = f(tx_1, tx_2)$$
$$= \begin{bmatrix} tx_2 \\ tx_1 - 2(tx_2) \end{bmatrix}$$
$$= \begin{bmatrix} tx_2 \\ t(x_1 - 2x_2) \end{bmatrix}$$
$$= t \begin{bmatrix} x_2 \\ x_1 - 2x_2 \end{bmatrix} = t f(x_1, x_2)$$

$\therefore$ (L2) holds.

Since f satisfies (L1) and (L2), f is linear!

$\blacksquare$

To show a function is NOT linear, we must exhibit specific vectors that violate (L1) or (L2).

Ex: Show that the function $g: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $g(x_1, x_2, x_3) = (x_1 x_2, x_3)$ is NOT linear.

Solution: Note that $g(1,1,1) = \begin{bmatrix} (1)(1) \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$g(2,2,2) = \begin{bmatrix} (2)(2) \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Uh oh...

$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ but $g(2,2,2) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2g(1,1,1)$

$\therefore$ (L2) fails, so g is NOT linear!

Alternative argument:

$g(1,0,0) = \begin{bmatrix} (1)(0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $g(0,1,0) = \begin{bmatrix} (0)(1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

but $g(1,1,0) = \begin{bmatrix} (1)(1) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, but

$$g(1,1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = g(1,0,0) + g(0,1,0).$$

Thus, (L1) fails, so g is NOT linear.

Ex: Is $h: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$h(x_1, x_2, x_3) = (x_1 - x_2,\ x_1 + 4x_3)$$

a linear map?

Solution: Let's try to prove

(L1) $h(\vec{x} + \vec{y}) = h(\vec{x}) + h(\vec{y})$ and (L2) $h(t\vec{x}) = t\,h(\vec{x})$

for all $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

(L1):
$$h(\vec{x}+\vec{y}) = h\left(x_1+y_1,\ x_2+y_2,\ x_3+y_3\right)$$

$$= \begin{bmatrix} (x_1+y_1) - (x_2+y_2) \\ (x_1+y_1) + 4(x_3+y_3) \end{bmatrix}$$

$$= \begin{bmatrix} (x_1 - x_2) + (y_1 - y_2) \\ (x_1 + 4x_3) + (y_1 + 4y_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_2 \\ x_1 + 4x_3 \end{bmatrix} + \begin{bmatrix} y_1 - y_2 \\ y_1 + 4y_3 \end{bmatrix} = h(\vec{x}) + h(\vec{y}).$$

$\therefore$ (L1) holds.

(L2) $h(t\vec{x}) = h(tx_1, tx_2, tx_3)$

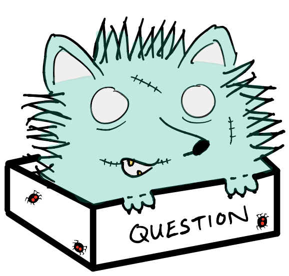
$$= \begin{bmatrix} tx_1 - tx_2 \\ tx_1 + 4(tx_3) \end{bmatrix}$$

$$= \begin{bmatrix} t(x_1 - x_2) \\ t(x_1 + 4x_3) \end{bmatrix}$$

$$= t\begin{bmatrix} x_1 - x_2 \\ x_1 + 4x_3 \end{bmatrix} = t\,h(\vec{x})$$

$\therefore$ (L2) holds

Since h satisfies (L1) and (L2), h is linear.



We know that some linear mappings (matrix mappings) can be described using matrices. Can matrices be used to describe all linear mappings?

Hmm... to answer this let's consider an example.

Ex: Consider the mapping $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$f(x_1, x_2) = (x_2, x_1 - x_2),$$

which we showed above is linear.

If this were a matrix mapping f_A, then its columns should be...

$$1^{st} \text{ column} = f(\vec{e}_1) = f(1,0) = (0,1)$$

$$2^{nd} \text{ column} = f(\vec{e}_2) = f(0,1) = (1,-1).$$

Taking $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ we get

$$\boxed{f_A(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - x_2 \end{bmatrix} = f(x_1, x_2)}$$

So yes, f is a matrix mapping! It turns out this is always the case.

Theorem: If $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear, then L can be represented as a matrix mapping with matrix

$$[L] = \left[L(\vec{e}_1) \; L(\vec{e}_2) \; \cdots \; L(\vec{e}_n) \right]$$

Proof: $L(\vec{x}) = L(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$

$$= x_1 L(\vec{e}_1) + x_2 L(\vec{e}_2) + \cdots + x_n L(\vec{e}_n)$$

$$= \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) & \cdots & L(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= [L]\vec{x}$$

$\blacksquare$

Ex: Above, we saw that $h: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$h(x_1, x_2, x_3) = (x_1 - x_2,\ x_1 + 4x_3)$$

is linear. How can we represent this as a matrix mapping? Use this to find $h(2,1,0)$.

Solution: The corresponding matrix is

$$[h] = \begin{bmatrix} h(\vec{e}_1) & h(\vec{e}_2) & h(\vec{e}_3) \end{bmatrix}$$

Since

$$h(\vec{e}_1) = h(1,0,0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$h(\vec{e}_2) = h(0,1,0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \text{and}$$

$$h(\vec{e}_3) = h(0,0,1) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}; \quad \text{we have}$$

$$\boxed{[h] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}.}$$

Let's find $h(2,1,0)$!

$$h(2,1,0) = [h]\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}.$$

Combinations of Mappings

Suppose we have functions

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $M : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, and $t \in \mathbb{R}$.

From these functions, we can make new functions:

[Sum] $L+M : \mathbb{R}^n \to \mathbb{R}^m$ given by $(L+M)(\vec{x}) = L(\vec{x}) + M(\vec{x})$

[Multiple] $tL : \mathbb{R}^n \to \mathbb{R}^m$ given by $(tL)(\vec{x}) = tL(\vec{x})$,

FACT: If L and M are linear, then so are $L+M$ and tL!

[This is not a tough fact to prove, but it is tedious. Try it as an exercise!]

Note: If L and M are linear, we can find $[L]$ and $[M]$ by calculating $L(\vec{e_i})$ and $M(\vec{e_i})$.

Can we use $[L]$ and $[M]$ to find $[L+M]$?

Can we use $[L]$ to find $[tL]$?

Yes! It turns out that $\boxed{[L+M] = [L] + [M]}$

and $\boxed{[tL] = t[L]}$.

Why? Well, for any $\vec{x} \in \mathbb{R}^n$, we have

$$
\begin{aligned}
[L+M]\vec{x} &= (L+M)\vec{x} \\
&= L(\vec{x}) + M(\vec{x}) \\
&= [L]\vec{x} + M[\vec{x}] = \big([L]+[M]\big)\vec{x}
\end{aligned}
$$

$\therefore$ $[L+M] = [L] + [M]$ by Q3 of Assignment 7.

Exercise: Imitate the above argument to show that

$$[tL] = t[L].$$

Ex: Consider the mappings $L(x_1, x_2) = (x_1, x_1 + x_2)$

$$M(x_1, x_2) = (x_2, x_1 - x_2).$$

(As an exercise, show that L and M are linear.)

By the above FACT, the functions

$$(L+M)(x_1, x_2) = L(x_1, x_2) + M(x_1, x_2)$$

$$= \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \end{bmatrix}$$

and

$$(2L)(x_1, x_2) = 2L(x_1, x_2)$$

$$= 2\begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_1 + 2x_2 \end{bmatrix}$$

are also linear.

What are $[L]$, $[M]$, $[L+M]$, $[2L]$?

We have $[L] = \left[L(\vec{e}_1) \;\; L(\vec{e}_2) \right] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$[M] = \left[M(\vec{e}_1) \;\; M(\vec{e}_2) \right] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

We could do the same thing to find $[L+M]$ and $[2L]$,

OR we could use the equations

$$[L+M] = [L] + [M] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$[2L] = 2[L] = 2\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

Compositions

Suppose we have mappings $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $M : \mathbb{R}^m \longrightarrow \mathbb{R}^p$. We can form the composition

$$\underbrace{M \circ L :}\; \mathbb{R}^n \longrightarrow \mathbb{R}^p \text{ given by } (M \circ L)(\vec{x}) = M(L(\vec{x}))$$

We read this right to left.
First apply L, then apply M

Pictorially ...

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "L"] \arrow[rr, bend left, "M \circ L"] & \mathbb{R}^m \arrow[r, "M"] & \mathbb{R}^p
\end{tikzcd}
$$

Just like in the cases of addition and scalar multiplication, we have ...

$$\boxed{\text{If } L \text{ and } M \text{ are linear, then so is } M \circ L}$$

Why? Well, if $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then

$$
\begin{aligned}
(M \circ L)(\vec{x} + \vec{y}) &= M(L(\vec{x} + \vec{y})) \\
&= M(L(\vec{x}) + L(\vec{y})) && (\text{as } L \text{ is linear}) \\
&= M(L(\vec{x})) + M(L(\vec{y})) && (\text{as } M \text{ is linear}) \\
&= (M \circ L)(\vec{x}) + (M \circ L)(\vec{y}) && \therefore \text{(L1) holds}
\end{aligned}
$$

$$
\begin{aligned}
(M \circ L)(t\vec{x}) &= M(L(t\vec{x})) \\
&= M(t L(\vec{x})) && (\text{as } L \text{ is linear})
\end{aligned}
$$

$$= t\,M(L(\vec{x})) \qquad \text{(as } M \text{ is linear)}$$

$$= t\,(M \circ L)(\vec{x}) \qquad \therefore \text{(L2) holds}$$

Just like before, if L and M are linear and we know $[L]$ and $[M]$, then we can find $[M \circ L]$.

Indeed, if $\vec{x}$ is any vector in $\mathbb{R}^n$, then

$$[M \circ L]\vec{x} = (M \circ L)(\vec{x})$$

$$= M(L(\vec{x}))$$

$$= M([L]\vec{x}) = [M][L]\vec{x}.$$

By Q3 of Assignment 7, we have $\boxed{[M \circ L] = [M][L].}$

Ex: Consider the linear mappings from the previous example:

$$L(x_1, x_2) = (x_1, x_1 + x_2), \quad M(x_1, x_2) = (x_2, x_1 - x_2).$$

We found that $[L] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $[M] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

By the above results, the composition map

$$(M \circ L)(x_1, x_2) = M(L(x_1, x_2))$$
$$= M(x_1, x_1 + x_2) = (x_1 + x_2, -x_2)$$

is also linear.

<u>What is $[M \circ L]$?</u>

We could find this by computing $(M \circ L)(\vec{e_1})$ and $(M \circ L)(\vec{e_2})$ ... <u>OR</u> we could use

$$[M \circ L] = [M][L] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}$$