§2.3 – Applications to Spanning ; Linear Independence

Consider once again our first example from §2.1 :

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - 5x_3 &= 7 \end{aligned} \quad \longleftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -5 & 7 \end{array}\right]$$

This system can be written in a new way using vectors :

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

columns of coefficient matrix

Our solution was $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and sure enough...

$$0 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} !$$

So...

Finding solutions to $[A \mid \vec{b}]$

Finding points on the intersection of hyperplanes. ⟷ Finding linear combinations of the columns of A that equal $\vec{b}$.

(Row Picture) (Column Picture)

This new perspective can help us solve some (formerly) tricky problems from Chapter 1.

① Spanning Problems

Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a set of vectors in $\mathbb{R}^m$, and let

$$A = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$$

be the matrix whose columns are $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

> A vector $\vec{b} \in \mathbb{R}^m$ belongs to Span$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ if and only if the system $[A \mid \vec{b}]$ is consistent.

If it is consistent, any solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ gives a linear combination that equals $\vec{b}$:

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n = \vec{b}.$$

Ex: Does $\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ belong to Span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \right\}$?

Solution: Here $A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & -1 & 1 & -3 \\ 1 & 5 & 4 & 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$.

Is $[A \mid \vec{b}]$ consistent?

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & -2 \\ 1 & -1 & 1 & -3 & -3 \\ 1 & 5 & 4 & 3 & 1 \end{array}\right] \underset{\substack{R_2 - R_1 \\ R_3 - R_1}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & -2 \\ 0 & -2 & -1 & -2 & -1 \\ 0 & 4 & 2 & 4 & 3 \end{array}\right] \underset{R_3 + 2R_2}{\sim}$$

$$\left[\begin{array}{cccc|c} \underline{1} & 1 & 2 & -1 & -2 \\ 0 & \underline{-2} & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & \underline{1} \end{array}\right] \quad \text{(REF)}$$

The system is inconsistent, so $\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ is NOT in the span!

Ex: Does $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ belong to $\text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$?

Solution: As an exercise, check that

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 3 & 7 & 1 & 1 & 2 \\ 4 & 7 & -1 & 4 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} \underline{1} & 0 & -2 & 0 & -2 \\ 0 & \underline{1} & 1 & 0 & 1 \\ 0 & 0 & 0 & \underline{1} & 1 \end{array}\right] \quad \text{(RREF)}$$

This system is consistent, so $\begin{bmatrix}1\\2\\3\end{bmatrix}$ does belong to this Span! The solution to the system is

$$\vec{x} = \begin{bmatrix}-2\\1\\0\\1\end{bmatrix} + t\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}, \quad t \in R$$

So if $\vec{x} = \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$ (or any other vector in the solution)

we have $$(-2)\begin{bmatrix}1\\3\\4\end{bmatrix} + (1)\begin{bmatrix}2\\7\\7\end{bmatrix} + (0)\begin{bmatrix}0\\1\\-1\end{bmatrix} + (1)\begin{bmatrix}1\\1\\4\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Ex: Show that any vector in $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\end{bmatrix}$, and $\begin{bmatrix}0\\1\end{bmatrix}$.

[This is Q4(a) of Assignment 4]

Solution: We need to show that the system

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & \multirow{2}{*}{$\vec{b}$}\\1 & 2 & 1 & \end{array}\right]$$ is consistent for all $\vec{b}$.

But ... $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \underset{R_2 - R_1}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \overset{R_1 - R_2}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ (RREF)

The rank of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ is 2, so every row has a leading 1; the system is always consistent!

$\therefore$ Every $\vec{b} \in \mathbb{R}^2$ belongs to $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

Note: In §2.2 we saw that $[A \mid \vec{b}]$ is consistent for all vectors $\vec{b}$ if and only if $\text{Rank}(A) = m$ (# of equations)

In terms of spanning:

$$\boxed{\text{The columns of } A \text{ span } \mathbb{R}^m \text{ if and only if } \text{Rank}(A) = m}$$

An interesting consequence: $\boxed{\text{If } \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \mathbb{R}^m, \text{ then } n \geq m.}$

Why?

If $\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \mathbb{R}^m$, then $\text{Rank}(A) = m$ (by above)

But then $m = \text{Rank}(A) \leq \min(m,n) \leq n$

so $n \geq m$! ∎

Ex: Is $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \\ 10 \end{bmatrix}, \begin{bmatrix} \pi \\ \pi^2 \\ \pi^3 \\ \pi^4 \\ \pi^5 \end{bmatrix} \right\}$ a spanning set for $\mathbb{R}^5$?

Solution: NO WAY! By the above result, we need at least 5 vectors to span $\mathbb{R}^5$!

② Linear Independence Problems

Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a set of vectors in $\mathbb{R}^m$ and let

$$A = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$$

be the matrix whose columns are $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

$\boxed{\text{The columns of } A \text{ are linearly independent if and only if the only solution to the homogeneous system } [A \mid \vec{0}] \text{ is } \vec{x} = \vec{0}.}$

So $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent if and only if $[A \mid \vec{0}]$ has a unique solution.

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Solution: The homogeneous system is

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \underset{R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underset{R_3 - R_2}{\sim}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1 + R_3}{\sim} \begin{bmatrix} \underline{1} & 0 & 0 \\ 0 & \underline{1} & 0 \\ 0 & 0 & \underline{1} \end{bmatrix} \text{(RREF)}$$

(Remember, the right hand side is $\vec{0}$)

The only solution is $\vec{x} = \vec{0}$.

$\therefore$ The set is linearly independent.

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}$ linearly independent?

Solution: The homogeneous system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \underline{1} & 0 & 3 \\ 0 & \underline{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

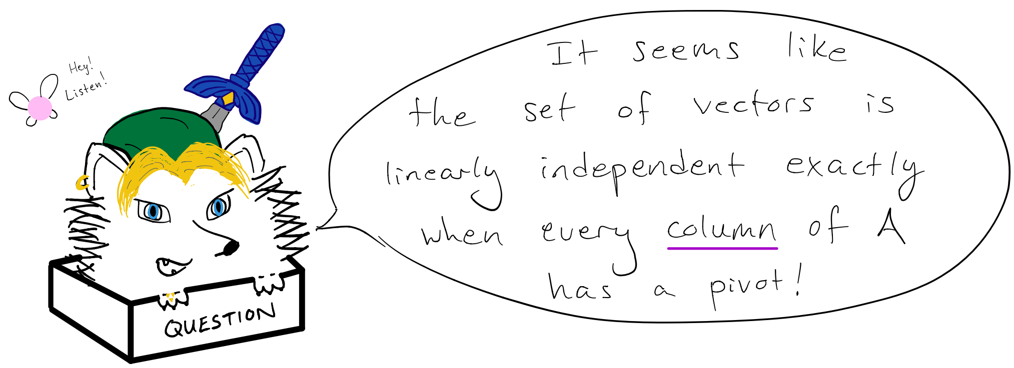
The rank is 2, so

# of parameters = # of columns − Rank = 3 − 2 = 1.

There are infinitely many solutions, so the set is linearly dependent.

The solution is $\vec{x} = t\begin{bmatrix}-3\\2\\1\end{bmatrix}$, $t \in \mathbb{R}$, so any such vector gives us a linear combination:

$$-3\begin{bmatrix}1\\2\\0\\0\end{bmatrix} + 2\begin{bmatrix}1\\3\\1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\-2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}!$$



That's exactly right!

From §2.2: a consistent system has $n - \text{Rank}(A)$ parameters, so $[A \mid \vec{0}]$ has a unique solution if and only if $n - \text{Rank}(A) = 0$.

[i.e., $\text{Rank}(A) = n$, or equivalently, every column has a pivot.]

In terms of linear independence:

$$\boxed{\text{The columns of } A = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \text{ are linearly independent if and only if } \operatorname{Rank}(A) = n.}$$

An interesting consequence:

$$\boxed{\text{If } \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \text{ is a linearly independent set of vectors in } \mathbb{R}^m, \text{ then } n \leq m.}$$

Why?

Because if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent, then $\operatorname{Rank}(A) = n$ (by above). But then

$$\underline{n = \operatorname{Rank}(A) \leq \min(m,n) \leq m},$$

so $n \leq m$! ∎

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \right\}$ linearly independent?

Solution: No WAY! By the above result, a linearly independent set in $\mathbb{R}^3$ has at most 3 vectors!

③ Basis Problems

To show that a set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^m$, we must show that

(i) $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent (so $\underline{A = [\vec{v}_1 \cdots \vec{v}_n] \text{ has rank} = n}$)

(ii) $\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \mathbb{R}^m$ (so $\underline{A = [\vec{v}_1 \cdots \vec{v}_n] \text{ has rank} = m}$)

Note: This is only possible if $\underline{\underline{m = n}}$

[# of rows = # of columns]

$$\boxed{\text{The columns of } A = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n] \text{ are a basis for } \mathbb{R}^n \text{ if and only if } \text{Rank}(A) = n.}$$

Consequence: $\boxed{\text{Every basis for } \mathbb{R}^n \text{ has exactly } n \text{ vectors!}}$

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Solution: We have that

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \underset{\substack{R_2-R_1 \\ R_3-3R_1}}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{bmatrix} \underset{R_3-4R_2}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \underset{R_3\cdot(\frac{-1}{2})}{\sim}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2-R_3}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1+R_2}{\sim} \begin{bmatrix} \underline{1} & 0 & 0 \\ 0 & \underline{1} & 0 \\ 0 & 0 & \underline{1} \end{bmatrix}$$

So Rank $= 3 =$ # of columns (the set is linearly independent)

$=$ # of rows (the set spans $\mathbb{R}^3$)

$$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$ is a basis for $\mathbb{R}^3$!

Theorem [Big Theorem, Version 1]

Let A be a matrix with n columns and n rows.

The following are equivalent:

(1) The columns of A are linearly independent;

(2) The columns of A span $\mathbb{R}^n$;

(3) The columns of A are a basis for $\mathbb{R}^n$;

(4) $\text{Rank}(A) = n$;

(5) $[A \mid \vec{0}]$ has a unique solution;

(6) $[A \mid \vec{b}]$ is consistent for all $\vec{b} \in \mathbb{R}^n$.

With a bit of work (see text) one can extend these ideas to prove the following:

If S is any subspace of $\mathbb{R}^n$, then any two bases for S have the same number of vectors

Definition: If S is a subspace of $\mathbb{R}^n$ with a basis containing k vectors, we say that the dimension of S is k:

$$\boxed{\dim S = k}$$

Ex: The standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ (see §1.7) is a basis for $\mathbb{R}^n$ with n vectors, so $\dim \mathbb{R}^n = n$

Ex: From §1.7, a line in $\mathbb{R}^n$ has dimension 1

a plane in $\mathbb{R}^n$ has dimension 2

a hyperplane in $\mathbb{R}^n$ has dimension $n-1$.