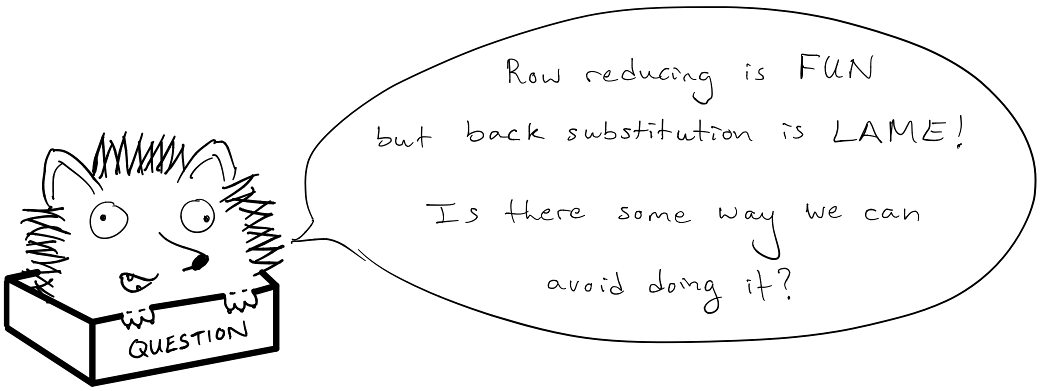
§2.2 – Gauss-Jordan Elimination



Yes! Consider our first example from §2.1:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -5 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \text{(REF)}$$

By doing a few more row operations, we can drastically simplify the back substitution process:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \underset{(-1)R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 + 2R_3 \\ R_2 - R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \overset{R_1 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

From here we quickly get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

<u>No back substitution!</u>

The reason it worked? The matrix is in a <u>very</u> special type of REF, called <u>reduced row echelon form</u>

Definition: A matrix is said to be in **reduced row echelon form** (<u>RREF</u>) if

(1) it is in REF,

(2) all leading entries are 1 (called <u>leading 1's</u>), and

(3) in any column with a leading 1, all other entries are 0.

Ex: $\begin{bmatrix} \underline{1} & 0 & 0 & 3 \\ 0 & 0 & \underline{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in RREF, but $\begin{bmatrix} \underline{1} & 0 & 0 \\ 0 & \underline{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{1} & 0 \\ 0 & 0 & \underline{1} \end{bmatrix}$ are NOT (they fail (2) and (3), respectively)

We can always put a matrix into RREF by first writing it in REF, and then eliminating the entries above the leading 1's from right to left.

This process is called **Gauss-Jordan elimination.**

Ex: Write $\left[\begin{array}{cccc|c} 1 & 3 & 5 & 8 & -1 \\ 2 & 6 & 9 & 14 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ in RREF, and find the general solution.

Solution:

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 8 & -1 \\ 2 & 6 & 9 & 14 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{R_2-2R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 3 & 5 & 8 & -1 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{R_2(-1)}{\sim}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 8 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{\substack{R_1-8R_3 \\ R_2-2R_3}}{\sim} \left[\begin{array}{cccc|c} 1 & 3 & 5 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{R_1-5R_2}{\sim}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \text{ (RREF)}$$

Now it's easy to read off the solution.

Since x_2 is not leading, it is free:

$$x_2 = t, \ t \in \mathbb{R}$$

Then
$$x_4 = 0$$
$$x_3 = -2$$
$$x_1 = 9 - 3x_2 = 9 - 3t$$

So the solution is

$$\boxed{\vec{x} = \begin{bmatrix} 9-3t \\ t \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ t \in \mathbb{R}}$$

One important thing about RREF: it's <u>unique</u>.

$$\boxed{\text{A matrix has one and only one RREF!}}$$

This is NOT the case for REF.

The Rank of a Matrix

From 2.1, we know that the number of leading entries in a REF can tell us if a system is

(a) inconsistent;

(b) consistent with a unique solution; or

(c) consistent with infinitely many solutions.

Let's give this number a name: the rank!

Definition: The rank of a matrix is the number of leading 1's in its RREF.

Note: The rank of a matrix A is actually the number of leading entries in any REF of A.

(But since the RREF is unique and the REF is not, it's easier to make arguments in terms of RREF)

Ex: $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right]$ has rank 2

(RREF)

$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$ has rank 2.

(REF)

What's the rank of $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$? We have

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \underset{R_2 - 2R_1}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ (RREF) So rank = 1

Note: If A is a matrix with m rows and n columns, then $\text{Rank}(A) \leq \min(m,n)$. (why?)

Our findings from §2.1 can be summarized in terms of the rank of a matrix:

Theorem: Let $[A \mid \vec{b}]$ be a system of m equations in n variables.

(1) The system is consistent if and only if the rank of A = the rank of $[A \mid \vec{b}]$.

(2) If the system is consistent, the number of parameters in the general solution is the number of variables minus the rank:

$$\boxed{\#\text{ of parameters} = n - \text{Rank}(A)}$$

From these two points we can make some very interesting observations...

Corollary Let $[A \mid \vec{b}]$ be a system of m equations in n variables.

(1) The system is consistent for all vectors $\vec{b} \in \mathbb{R}^m$ if and only if $\boxed{\text{Rank}(A) = m}$

(i.e., every row of A has a leading 1!)

(2) If the system is consistent and has a unique solution for all $\vec{b} \in \mathbb{R}^m$, then $\boxed{\text{Rank}(A) = n = m}$

Homogeneous Systems

In the next section and throughout the remainder of the course, we will often deal with systems whose right hand side is $\vec{0}$.

Such a system is called homogeneous.

Ex:
$$\begin{aligned} 2x_1 + x_2 \quad\;\; &= 0 \\ x_1 + x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0 \end{aligned}$$
is a homogeneous system

Note:

(1) Our ERO's don't change the right hand side of a homogeneous system, so we often just omit it.

So instead of writing $\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$ for the above system, we write $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and remember that the right hand side is $\vec{0}$.

(2) A homogeneous system is <u>always consistent</u>:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ is always a solution!

Since we know a solution exists, we are often more interested in the <u>number of parameters</u> in the general solution

Ex:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \underset{\sim}{R_1 \updownarrow R_2} \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \underset{R_2 - 2R_1}{\sim}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \sim \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} (-1)R_2 \\ \sim \end{matrix}$$

$$\begin{bmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{1} & -2 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

<u>Rank is 2</u>

<u># of parameters = 1</u>

<u>General Solution:</u> $\vec{x} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$