Chapter 2 – Systems of Linear Equations

§2.1 – Gaussian Elimination

In this chapter we'll be looking at linear equations!

Ex: $x_1 + 3x_2 = 8 \quad (1)$

Ex: $2x_1 - x_2 = -5 \quad (2)$

In general: $\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$

where $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$.

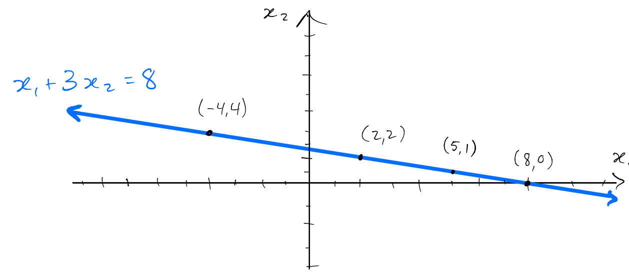
In particular, we'll be interested in finding Solutions to these equations!

Ex: $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is a solution to $x_1 + 3x_2 = 8$,

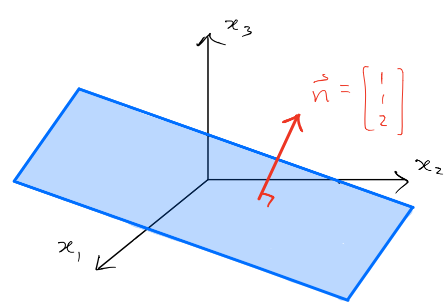
because $(5) + 3(1) = 8$.

Check: $\begin{bmatrix} 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 8/3 \end{bmatrix}$ are solutions as well.

Solutions to $x_1 + 3x_2 = 8$ form a line in $\mathbb{R}^2$:



Solutions to $x_1 + x_2 + 2x_3 = 4$ form a plane in $\mathbb{R}^3$:



In general, the solutions to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ form a hyperplane in $\mathbb{R}^n$.

Instead of finding the solution to just one linear equation, we'll be interested in solving several equations simultaneously!

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{matrix} \quad = \quad \text{System of linear equations}$$

m rows $\longleftrightarrow$ m equations

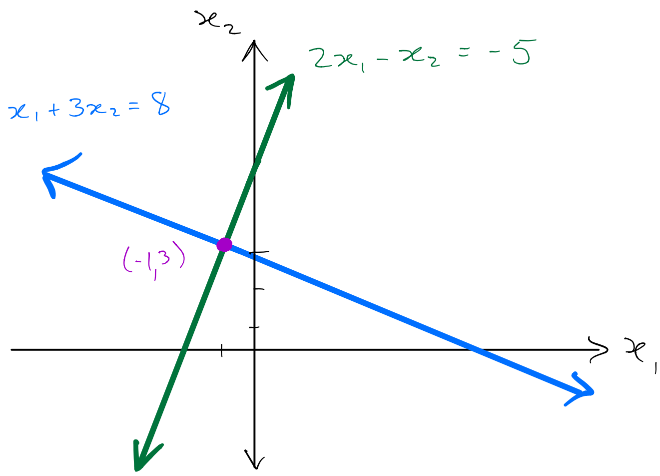
n columns $\longleftrightarrow$ n unknowns

a_{ij} = coefficient of x_j in i^{th} equation.

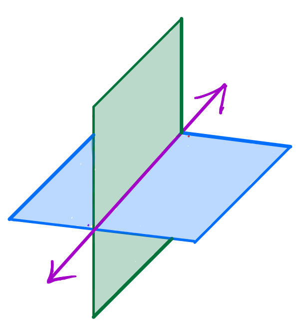
Since each row is a hyperplane in $\mathbb{R}^n$

$$\boxed{\begin{array}{ccc}\text{Solutions to system} & = & \text{Points in intersection} \\ \text{of linear equations} & & \text{of hyperplanes}\end{array}}$$

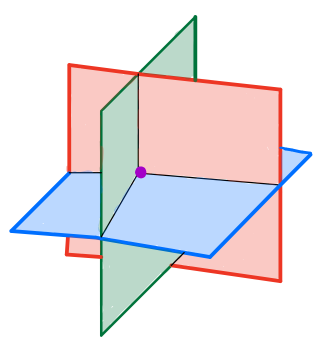
In $\mathbb{R}^2$, this is an intersection of lines:



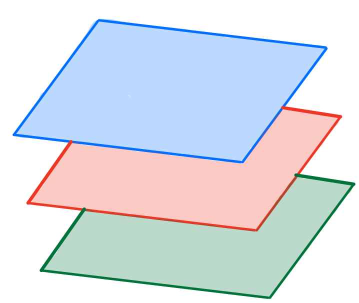
In $\mathbb{R}^3$, this is an intersection of planes:



(Solution is a line)



(Solution is a point)



(No solution)

Solving a System : Gaussian Elimination

Consider the system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - 5x_3 &= 7 \end{aligned}$$

We will solve this system by eliminating some of the variables from some of the equations without changing the solution set!

Two systems with the same solution set will be called equivalent (denoted by $\sim$)

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \quad (R_1) \\ 2x_2 + x_3 &= 3 \quad (R_2) \\ 2x_1 + x_2 - 5x_3 &= 7 \quad (R_3) \end{aligned}$$

$\sim$ Add $(-2)R_1$ to R_2

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \quad (R_1) \\ 2x_2 + x_3 &= 3 \quad (R_2) \\ -x_2 - x_3 &= -1 \quad (R_3) \end{aligned}$$

Any solution to the first system is also a solution to the second.

Since this operation can be reversed (by adding R_1 to R_2), any solution to the second system is also a solution to the first.

$\therefore$ The systems are equivalent.

$\sim$ Swap R_2 and R_3

Reordering equations does not change solution set.

$\therefore$ These systems are equivalent.

$$
\begin{aligned}
x_1 + x_2 - 2x_3 &= 4 \qquad (R_1)\\
-x_2 - x_3 &= -1 \qquad (R_2)\\
2x_2 + x_3 &= 3 \qquad (R_3)
\end{aligned}
$$

$\sim$ Multiply R_2 by (-1)

This step is reversible and doesn't change solution set

$\therefore$ These systems are equivalent.

$$
\begin{aligned}
x_1 + x_2 - 2x_3 &= 4 \qquad (R_1)\\
x_2 + x_3 &= 1 \qquad (R_2)\\
2x_2 + x_3 &= 3 \qquad (R_3)
\end{aligned}
$$

$\sim$ Add $(-2)R_2$ to R_3

Same as above...

These systems are equivalent.

$$
\begin{aligned}
x_1 + x_2 - 2x_3 &= 4 \qquad (R_1)\\
x_2 + x_3 &= 1 \qquad (R_2)\\
-x_3 &= 1 \qquad (R_3)
\end{aligned}
$$

From R_3: $\underline{x_3 = -1}$.

From R_2: $x_2 + x_3 = 1 \Rightarrow x_2 + (-1) = 1$

$\Rightarrow \underline{x_2 = 2}$

From R_1: $x_1 + x_2 - 2x_3 = 4 \Rightarrow x_1 + (2) - 2(-1) = 4$

$\Rightarrow \underline{x_1 = 0}$

Since the final system is equivalent to the original, our solution is $\boxed{\vec{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.}$ (check!)

The final system,

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ x_2 + x_3 &= 1 \\ -x_3 &= 1 \end{aligned},$$

has a very special form which makes it easy to solve.

Definition: A system of linear equations is said to be in row echelon form (REF) if the leading variable in every row (the first variable with non-zero coefficient) is strictly to the right of the leading variable in the row above.

Leading variables on lower rows are strictly to the right:

$$\begin{aligned} \boxed{x_1} + x_2 - 2x_3 &= 4 \\ \boxed{x_2} + x_3 &= 1 \\ -\boxed{x_3} &= 1 \end{aligned}$$

Fact: Every system can be put into REF using a sequence of elementary row operations (EROs).

EROs: (1) Multiply an equation by a non-zero constant
(2) Swap two equations
(3) Add a multiple of one equation to another equation

Using EROs to reduce a system to REF is called Gaussian Elimination or row reduction.

Solving for each x_i from the bottom up is called back substitution.

Ex: Using Gaussian elimination, find the solution set for

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \qquad (R_1)$$
$$2x_1 + 4x_2 + 7x_3 + 10x_4 = 25 \qquad (R_2)$$

Solution: Let's make the leading variable in R_2 occur to the right of the leading variable in R_1 (i.e., x_1)

To eliminate x_1 from R_2, add $(-2)R_1$ to R_2

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 10 \\ 2x_1 + 4x_2 + 7x_3 + 10x_4 &= 25 \end{aligned} \quad \underset{R_2 - 2R_1}{\sim} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 10 \\ x_3 + 2x_4 &= 5 \end{aligned}$$

(REF)

Note: Our two equations do not completely determine x_3 and x_4 : one can be chosen arbitrarily!

We always pick the non-leading variable to be the arbitrary one (here it's x_4). This is called a free variable!

x_2 is not a leading variable, so it's free too.

Since free variables can be chosen arbitrarily, we write

$$\underline{x_2 = s, \ s \in \mathbb{R}} \qquad \text{and} \qquad \underline{x_4 = t, \ t \in \mathbb{R}}$$

(s and t are called parameters.)

Now use back substitution to find the leading variables!

From R_2: $x_3 = 5 - 2x_4 = 5 - 2t$

From R_1:
$$\begin{aligned} x_1 &= 10 - 2x_2 - 3x_3 - 4x_4 \\ &= 10 - 2s - 3(5-2t) - 4t = -5 - 2s + 2t \end{aligned}$$

This means that the general solution to our system of equations is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5-2s+2t \\ s \\ 5-2t \\ t \end{bmatrix}, \quad s,t \in \mathbb{R}}$$

OR

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}}$$

Plane in $\mathbb{R}^4$ passing through $(-5,0,5,0)$!

Representing Systems with Matrices

Instead of writing $x_1, x_2, x_3, \ldots$ in a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We'll just record the coefficients a_{ij} and right-hand side $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ in a matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \text{the system's } \textbf{coefficient matrix}$$

$$[A \mid \vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] = \text{the system's } \textbf{augmented matrix}$$

Ex: The system

$$\begin{aligned} 3x_1 + 8x_2 - 18x_3 + x_4 &= 35 \\ x_2 - 3x_3 + x_4 &= -1 \\ x_1 + 2x_2 - 4x_3 &= 11 \end{aligned}$$

has coefficient matrix $A = \begin{bmatrix} 3 & 8 & -18 & 1 \\ 0 & 1 & -3 & 1 \\ 1 & 2 & -4 & 0 \end{bmatrix}$

and augmented matrix $[A \mid \vec{b}] = \left[\begin{array}{cccc|c} 3 & 8 & -18 & 1 & 35 \\ 0 & 1 & -3 & 1 & -1 \\ 1 & 2 & -4 & 0 & 11 \end{array}\right]$

Definition: A matrix is in **row echelon form** (REF) if

(1) All rows that are fully 0 occur at the bottom, and

(2) The first non-zero entry in a row, called the row's leading entry or pivot, is to the right of the leading entry from all higher rows.

Ex: $\begin{bmatrix} \underline{1} & 0 & 0 & 0 \\ 0 & 0 & \underline{6} & 8 \\ 0 & \underline{2} & 4 & 5 \end{bmatrix}$ is not in row echelon form because the leading entry in R_3 is NOT to the right of the leading entry in R_2.

$\begin{bmatrix} \underline{1} & 3 & 5 & 6 & 7 \\ \underline{2} & 4 & 5 & 8 & 8 \end{bmatrix}$ is not in REF because the leading entry of R_2 is beneath the leading entry of R_1, not to the right.

$\begin{bmatrix} \underline{1} & 1 \\ 0 & 0 \\ 0 & \underline{1} \end{bmatrix}$ is not in REF because there is a row of zeros above a non-zero row.

Both $\begin{bmatrix} \underline{1} & 1 & -1 & 3 & 0 \\ 0 & \underline{2} & 1 & 4 & 2 \\ 0 & 0 & 0 & \underline{5} & 3 \\ 0 & 0 & 0 & 0 & \underline{3} \end{bmatrix}$ and $\begin{bmatrix} 0 & \underline{1} & 0 & 1 \\ 0 & 0 & \underline{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are in REF.

Ex: Let's solve our first system:

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - 5x_3 &= 7 \end{aligned}$$

using matrices!

Put this next to row where operation occurs. Here it's in row 3

Swap R_2 and R_3

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -5 & 7 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & -1 & -1 & -1 \end{array}\right] \begin{array}{l} \\ R_2 \updownarrow R_3 \\ \\ \end{array}$$

Multiply R_2 by (-1)

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & -1 & -1 & -1 \\ 0 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ (-1)R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} \underline{1} & 1 & -2 & 4 \\ 0 & \underline{1} & 1 & 1 \\ 0 & 0 & \underline{-1} & 1 \end{array}\right] \Longrightarrow \begin{aligned} x_1 + x_2 - 2x_3 &= -4 \\ x_2 + x_3 &= 1 \\ -x_3 &= 1 \end{aligned}$$

With back substitution : $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}}$.

Ex: Find the general solution for

$$\begin{aligned} 3x_1 + 8x_2 - 18x_3 + x_4 &= 35 \\ x_2 - 3x_3 + x_4 &= -1 \\ x_1 + 2x_2 - 4x_3 \qquad &= 11 \end{aligned}$$

by row reducing the corresponding augmented matrix.

Solution: The augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & 8 & -18 & 1 & 35 \\ 0 & 1 & -3 & 1 & -1 \\ 1 & 2 & -4 & 0 & 11 \end{array}\right] \xrightarrow{R_3 \updownarrow R_1} \sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 1 & -3 & 1 & -1 \\ 3 & 8 & -18 & 1 & 35 \end{array}\right] R_3 - 3R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 2 & -6 & 1 & 2 \end{array}\right] R_3 - 2R_2 \sim \left[\begin{array}{cccc|c} \underline{1} & 2 & -4 & 0 & 11 \\ 0 & \underline{1} & -3 & 1 & -1 \\ 0 & 0 & 0 & \underline{-1} & 4 \end{array}\right] \text{(REF!)}$$

Going back to a system:

$$\begin{aligned} x_1 + 2x_2 - 4x_3 \quad\quad &= 11 \\ x_2 - 3x_3 + x_4 &= -1 \\ -x_4 &= 4 \end{aligned}$$

Since x_3 is not a leading variable, it is free:

$$\underline{x_3 = t, \ t \in \mathbb{R}}.$$

With back substitution we have

$$-x_4 = 4 \quad \Rightarrow \quad \underline{x_4 = -4}$$

$$x_2 - 3x_3 + x_4 = -1 \quad \Rightarrow \quad x_2 = -1 + 3x_3 - x_4$$
$$= -1 + 3t - (-4)$$
$$= \underline{3 + 3t}$$

$$x_1 + 2x_2 - 4x_3 = 11 \quad \Rightarrow \quad x_1 = 11 - 2x_2 + 4x_3$$
$$= 11 - 2(3+3t) + 4t$$
$$= \underline{5 - 2t}$$

The general solution: $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5-2t \\ 3+3t \\ t \\ -4 \end{bmatrix}, \quad t \in \mathbb{R}}$

OR

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}}$$

Ex: Find the general solution for

$$\begin{aligned} \tfrac{1}{2}x_1 + x_2 + \tfrac{1}{2}x_3 &= 4 \\ x_2 + 4x_3 &= 1 \\ x_1 + 3x_2 + 5x_3 &= 2 \end{aligned}$$

Solution: The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{2} & 1 & \frac{1}{2} & 4 \\ 0 & 1 & 4 & 1 \\ 1 & 3 & 5 & 2 \end{array}\right] \overset{2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 4 & 1 \\ 1 & 3 & 5 & 2 \end{array}\right] \underset{R_3 - R_1}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & -6 \end{array}\right] \underset{R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} \underline{1} & 2 & 1 & 8 \\ 0 & \underline{1} & 4 & 1 \\ 0 & 0 & 0 & \underline{-7} \end{array}\right]$$

(REF)

Going back to a system...

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 8 \\ x_2 + 4x_3 &= 1 \\ 0x_1 + 0x_2 + 0x_3 &= -7 \end{aligned}$$

(uh oh...)

Note: The left hand side of R_3 is 0. Since the right hand side of R_3 is non-zero, there is $\boxed{\text{no solution.}}$

A system with no solution is called **inconsistent.**

A system with at least one solution is called **consistent.**

Let's summarize what we saw in the examples:

Theorem: Suppose that $[A|\vec{b}]$ is the augmented matrix for a system of linear equations, and $[S|\vec{c}]$ is the REF of $[A|\vec{b}]$.

(1) The system is inconsistent if and only if some row of $[S|\vec{c}]$ has the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$$

where $c \neq 0$.

(2) If the system is consistent, then either

(a) # of pivots in S = # of variables, in which case there is a unique solution,

OR

(b) # of pivots in S < # of variables, in which case there are infinitely many solutions.