§1.7 – Spanning Sets ; Linear Independence (§1.2 in text)

Last time we proved that if $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then

$$S = \left\{ t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k \;\middle|\; t_1, t_2, \ldots, t_k \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^n$.

This subspace is <u>so</u> important that we give it a special name:

Definition: The subspace

$$S = \left\{ t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k \;\middle|\; t_1, t_2, \ldots, t_k \in \mathbb{R} \right\}$$

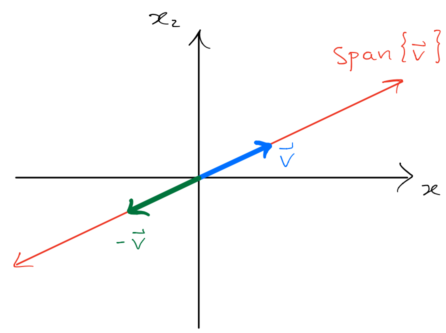
is called the subspace spanned by $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

We write

$$\boxed{S = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} = \operatorname{span} \mathcal{B}.}$$

We say that $\mathcal{B}$ is a spanning set for S, or that $\mathcal{B}$ spans S.

Ex: If $\vec{v}$ is a non-zero vector in $\mathbb{R}^2$, then



$$\mathcal{L} = \text{Span}\{\vec{v}\} = \{t\vec{v} : t \in \mathbb{R}\}$$

is a line through the origin. The set $\{\vec{v}\}$ is a spanning set for $\mathcal{L}$ (or $\{\vec{v}\}$ spans $\mathcal{L}$).

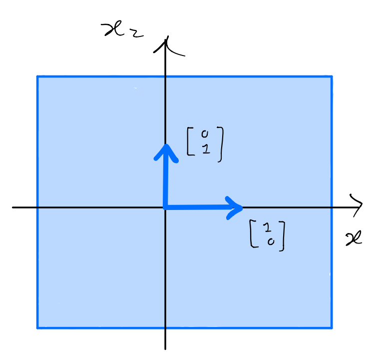
Note: The sets $\{-\vec{v}\}$, $\{2\vec{v}\}$, and $\{\frac{1}{3}\vec{v}\}$ are all spanning sets for $\mathcal{L}$ as well!

Q: What if instead we have two vectors $\vec{v}_1, \vec{v}_2$? What does $\text{span}\{\vec{v}_1, \vec{v}_2\}$ look like?

Ex: If $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

$$\text{span}\{\vec{v}_1, \vec{v}_2\} = \left\{ t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\}$$

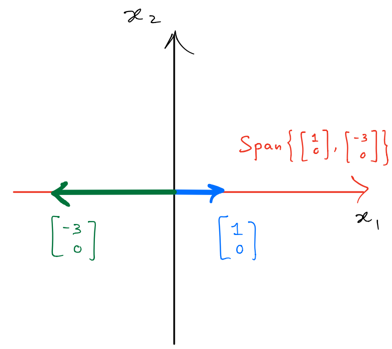
$$= \left\{ \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\} = \mathbb{R}^2$$



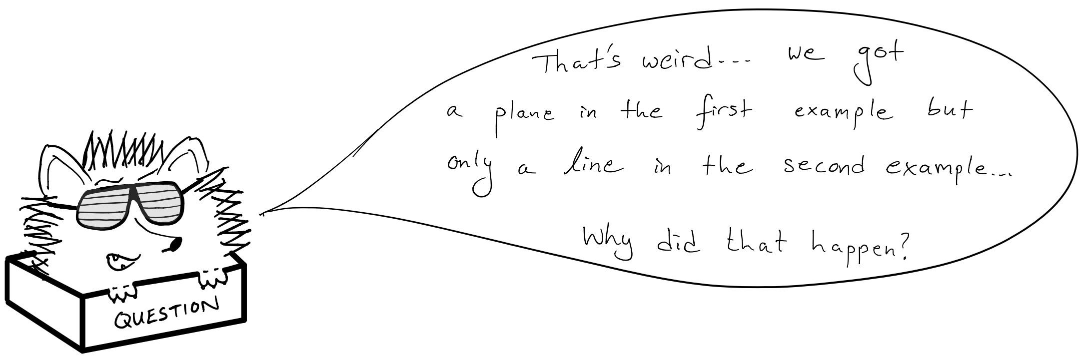
So $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$, the whole 2-dimensional plane

Ex: If $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$, then

$$\begin{aligned} \text{Span}\{\vec{v}_1, \vec{v}_2\} &= \left\{ t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\} \\ &= \left\{ (t_1 - 3t_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\} \\ &= \left\{ t \begin{bmatrix} 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} \end{aligned}$$



So $\text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, a *line* in $\mathbb{R}^2$.



Note: The vector $\vec{v}_2$ in the second example was a multiple of $\vec{v}_1$, so it didn't contribute anything to $\text{span}\{\vec{v}_1, \vec{v}_2\}$.

We found that

$$\text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{v}_1\},$$

so instead of spanning a plane, they only spanned a *line*.

Let's examine this in $\mathbb{R}^3$!

Ex: If $\vec{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} \sqrt{5} \\ -\sqrt{5} \\ 0 \end{bmatrix}$, then

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ t_1 \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} \sqrt{5} \\ -\sqrt{5} \\ 0 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}$$

$$= \left\{ (2t_1 - 3t_2 + \sqrt{5}\, t_3) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

So, $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Ex: If $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, then

$\vec{v}_2 = \vec{v}_1 + \vec{v}_3$, so

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ t_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}$$

$$= \left\{ t_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t_2 \left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right) + t_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}$$

$$= \left\{ (t_1 + t_2) \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + (t_2 + t_3) \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}.$$

$\Rightarrow \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

Theorem: Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ be vectors in $\mathbb{R}^n$.

If v_k can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}$ (i.e., $\vec{v}_k = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_{k-1}\vec{v}_{k-1}$), then

$$\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}\}$$

If there is a vector $\vec{v}_i \in \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ that can be written as a linear combination of the other vectors, then the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ will be called linearly dependent.

If this is not possible, we'll say that $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly independent.

Ex: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\end{bmatrix}\right\}$ is linearly dependent, as $\begin{bmatrix}-3\\0\end{bmatrix} = -3\begin{bmatrix}1\\0\end{bmatrix}$.

$\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}\right\}$ is linearly dependent, as $\begin{bmatrix}0\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\\3\end{bmatrix} + \begin{bmatrix}-1\\2\\0\end{bmatrix}$.

Note: If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly dependent, then there is a vector $\vec{v}_i$ and scalars $t_1, t_2, \dots, t_k \in \mathbb{R}$ such that

$$\vec{v}_i = t_1\vec{v}_1 + \cdots + t_{i-1}\vec{v}_{i-1} + t_{i+1}\vec{v}_{i+1} + \cdots + t_k\vec{v}_k$$

By moving $\vec{v}_i$ to the other side, we get

$$t_1\vec{v}_1 + \cdots + t_{i-1}\vec{v}_{i-1} - \vec{v}_i + t_{i+1}\vec{v}_{i+1} + \cdots + t_k\vec{v}_k = \vec{0}.$$

This coefficient $\neq 0$

So if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly dependent, then the equation

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}$$

has a solution with at least one $t_i \neq 0$.

This can <u>only</u> happen when $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly dependent!

<u>Why?</u>

Because if $t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}$ and some t_i is non-zero, then we can move $t_i\vec{v}_i$ over to get

$$-t_i\vec{v}_i = t_1\vec{v}_1 + \cdots + t_{i-1}\vec{v}_{i-1} + t_{i+1}\vec{v}_{i+1} + \cdots + t_k\vec{v}_k.$$

By dividing both sides by t_i, we have

$$\vec{v}_i = \left(\frac{-t_1}{t_i}\right)\vec{v}_1 + \cdots + \left(\frac{-t_{i-1}}{t_i}\right)\vec{v}_{i-1} + \left(\frac{-t_{i+1}}{t_i}\right)\vec{v}_{i+1} + \cdots + \left(\frac{-t_k}{t_i}\right)\vec{v}_k$$

$\Rightarrow$ $\vec{v}_i$ can be written as a linear combination of the other vectors in $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

$\Rightarrow$ $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly dependent.

This provides us with a more useful definition of linear dependence:

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is

(i) linearly dependent if there are scalars $t_1, t_2, \dots, t_k$ not all 0 such that

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}$$

(ii) linearly independent if the only solution to the equation

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}$$

is $t_1 = t_2 = \cdots = t_k = 0$ (called the trivial solution).

Ex: Is the set $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\-1\end{bmatrix}\right\}$ linearly independent?

Solution: Suppose t_1, t_2, t_3 are such that

$$t_1\begin{bmatrix}1\\0\\0\end{bmatrix} + t_2\begin{bmatrix}2\\1\\0\end{bmatrix} + t_3\begin{bmatrix}3\\2\\-1\end{bmatrix} = \vec{0}$$

This gives us a system of equations

$$\begin{cases} t_1 + 2t_2 + 3t_3 = 0 & (1) \\ \phantom{t_1 +{}} t_2 + 2t_3 = 0 & (2) \\ \phantom{t_1 + 2t_2 +{}} -t_3 = 0 & (3) \end{cases}$$

From ③ we get $\underline{t_3 = 0}$.

From ② we get $t_2 + \underbrace{2t_3}_{=0} = 0 \quad \Rightarrow \quad \underline{t_2 = 0}$.

From ① we get $t_1 + \underbrace{2t_2}_{=0} + \underbrace{3t_3}_{=0} = 0 \quad \Rightarrow \quad \underline{t_1 = 0}$.

Thus, the only solution to the equation

$$t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \vec{0}$$

is $t_1 = t_2 = t_3 = 0$ (the trivial solution).

$$\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ is linearly independent.}$$

Note: This example shows that deciding whether or not a set of vectors is linearly independent boils down to solving a system of equations. This is the topic of Chapter 2.

Exercise: Show that if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ contains $\vec{0}$, then this set is linearly dependent.

Bases:

The real idea behind today's lesson:

> the simplest spanning set for a subspace S is one that is linearly independent

Definition: If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a linearly independent set that spans a subspace S of $\mathbb{R}^n$, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is called a basis for S.

Ex: Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, and $S = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a spanning set for S, but it is not a basis for S. Why? Because

$$\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

So $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not linearly independent. By removing $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, we don't change the span:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}.$$

Since neither vector in $\left\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}$ is a multiple of the other, this set is linearly independent.

$$\therefore \left\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\} \text{ is a basis for } S.$$

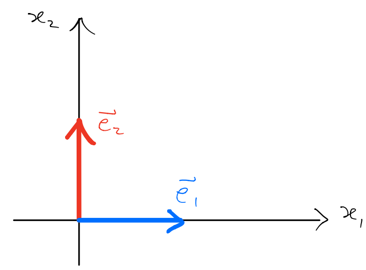
Definition: The standard basis for $\mathbb{R}^n$ is

$$\boxed{\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}}$$

where $\vec{e}_i$ has a 1 in the i^{th} entry and 0 elsewhere.

Ex: The standard basis for $\mathbb{R}^2$ is $\{\vec{e}_1, \vec{e}_2\}$, where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



$$\boxed{\text{Is } \{\vec{e}_1, \vec{e}_2\} \text{ really a basis for } \mathbb{R}^2\text{?}}$$

① It spans $\mathbb{R}^2$, as any vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

② It's linearly independent. Indeed, if $t_1\vec{e}_1 + t_2\vec{e}_2 = 0$, then

$$t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies t_1 = t_2 = 0.$$

$$\boxed{\text{So YES! It is a basis.}}$$

Exercise: Write down and draw the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ in $\mathbb{R}^3$. Show that $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is linearly independent and that $\operatorname{Span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \mathbb{R}^3$ (i.e., show it really is a basis for $\mathbb{R}^3$!)

Proposition: If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for a subspace S of $\mathbb{R}^n$, then every $\vec{x} \in S$ can be written as

$$\boxed{\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k}$$

in exactly one way.

Proof: We know it can be done in at least one way, since $\operatorname{Span}\mathcal{B} = S$.

But if $\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$ and $\vec{x} = s_1\vec{v}_1 + \cdots + s_k\vec{v}_k$

then $\vec{0} = \vec{x} - \vec{x} = \underbrace{(t_1 - s_1)}\vec{v}_1 + \cdots + \underbrace{(t_k - s_k)}\vec{v}_k$.

$= 0$ since $\mathcal{B}$ is linearly independent!

Thus $t_1 = s_1$, $t_2 = s_2$, $\ldots$, $t_k = s_k$

So there was really only one way to write $\vec{x}$! ∎

Lines, Planes, and Hyperplanes

Linear independence gives us a vector equation for lines, planes and hyperplanes in $\mathbb{R}^n$!

Definition:

(i) If $\vec{p}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$, then

$$\boxed{\vec{x} = \vec{p} + t\vec{v}} \qquad (t \in \mathbb{R})$$

is a line in $\mathbb{R}^n$ passing through $\vec{p}$.

(ii) If $\vec{p} \in \mathbb{R}^n$ and $\{\vec{v}_1, \vec{v}_2\} \subseteq \mathbb{R}^n$ is linearly independent, then

$$\boxed{\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2} \qquad (t_1, t_2 \in \mathbb{R})$$

is a plane in $\mathbb{R}^n$ passing through $\vec{p}$.

(iii) If $\vec{p} \in \mathbb{R}^n$ and $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}\} \subseteq \mathbb{R}^n$ is linearly independent, then

$$\boxed{\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_{n-1}\vec{v}_{n-1}} \quad (t_1, t_2, \dots, t_{n-1} \in \mathbb{R})$$

is a hyperplane in $\mathbb{R}^n$ passing through $\vec{p}$.

Ex: Does $S = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ describe a line, plane, or hyperplane in $\mathbb{R}^4$?

Solution: Since

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

these vectors form a linearly dependent set. So

$$S = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

This last set is linearly independent, as the vectors are not multiples of each other. Thus

$$S = \left\{ t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\}$$

linearly independent

$\therefore$ S is a plane in $\mathbb{R}^4$.