§1.6 – Subspaces of $\mathbb{R}^n$ (§1.2 in text)

We've been studying the geometry of vectors in $\mathbb{R}^n$ ($n=1,2,3,\dots$)

Everything we've done in $\mathbb{R}^n$ works because

(i) We can add vectors in $\mathbb{R}^n$ according to certain nice rules:

For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

(1) $\vec{x} + \vec{y} \in \mathbb{R}^n$ (closed under addition)
(2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (addition is commutative)
(3) $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ (addition is associative)
(4) There exists a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{z} + \vec{0} = \vec{z}$ for all $\vec{z} \in \mathbb{R}^n$ (zero vector)
(5) For each $\vec{x} \in \mathbb{R}^n$ there exists a vector $-\vec{x} \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverses)

(ii) We can multiply vectors in $\mathbb{R}^n$ by real scalars according to certain nice rules:

For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

(6) $t\vec{x} \in \mathbb{R}^n$ (closed under scalar multiplication)
(7) $s(t\vec{x}) = (st)\vec{x}$ (scalar multiplication is associative)
(8) $(s+t)\vec{x} = s\vec{x} + t\vec{x}$ (a distributive law)
(9) $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$ (another distributive law)
(10) $1\vec{x} = \vec{x}$ (scalar multiplicative identity)

Since $\mathbb{R}^n$ is closed under addition (1),
closed under scalar multiplication (6),

and these operations obey the other rules above,
we call $\mathbb{R}^n$ a vector space.

There are LOTS of other vector spaces out there...

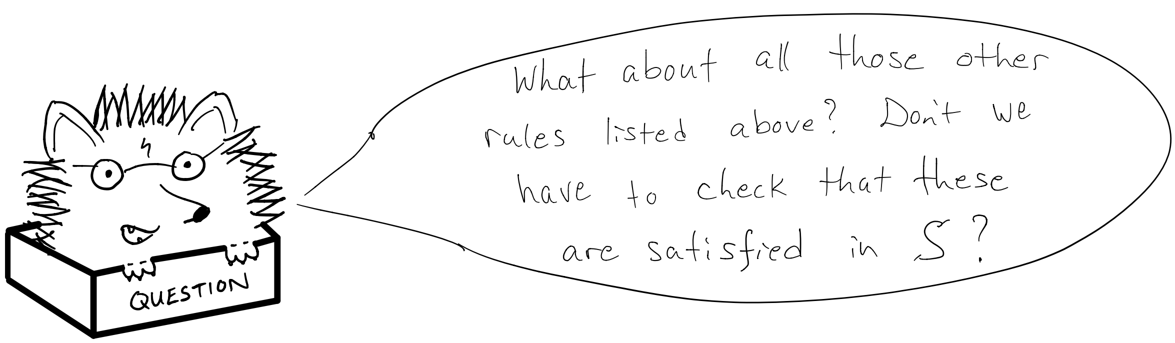
In this course we'll only be interested in vector spaces S within $\mathbb{R}^n$ (i.e., sets S contained in $\mathbb{R}^n$ that are

- closed under addition, and
- closed under scalar multiplication.)

Definition A non-empty subset S of $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if for all vectors $\vec{x}, \vec{y} \in S$ and $t \in \mathbb{R}$,

(1) $\vec{x} + \vec{y} \in S$ (S is closed under addition), and

(2) $t\vec{x} \in S$ (S is closed under scalar multiplication)



Thankfully, no! Any subspace of $\mathbb{R}^n$ will automatically inherits properties (2)-(5) and (7)-(10) from $\mathbb{R}^n$; so we don't need to check them!

Remarks: (i) In (2) of the definition of "subspace" above, we can set $t=0$ to see that

$$\boxed{\text{every subspace must contain } \vec{0}}$$

This is useful for showing that certain subsets of $\mathbb{R}^n$ are NOT subspaces.

For example, if $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + x_2 - 3x_3 = 5 \right\}$, then S is NOT a subspace of $\mathbb{R}^3$.

Why? $\vec{0} \notin S$!

In fact, if S is any line or plane in $\mathbb{R}^n$ that doesn't pass through the origin, then S is NOT a subspace!

(ii) The smallest subspace of $\mathbb{R}^n$ is $\{\vec{0}\}$.

This is sometimes called the trivial subspace.

(iii) The largest subspace of $\mathbb{R}^n$ is ... $\mathbb{R}^n$!

Ex: Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + 2x_2 + 3x_3 = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

Solution: There are 3 things to check.

(i) S is non-empty

We'll check that $\vec{0} \in S$. If $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, then $x_1 + 2x_2 + 3x_3 = 0 + 2(0) + 3(0) = 0$. So $\vec{0} \in S$.

$\therefore$ S is non-empty.

(ii) S is closed under addition

Suppose that $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in S$, so

$x_1 + 2x_2 + 3x_3 = 0$ and $y_1 + 2y_2 + 3y_3 = 0$.

We need to check that $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ belongs to S.

We have

$$
\begin{aligned}
(x_1+y_1) + 2(x_2+y_2) + 3(x_3+y_3) &= \underbrace{(x_1 + 2x_2 + 3x_3)}_{=0} + \underbrace{(y_1 + 2y_2 + 3y_3)}_{=0} \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

$\therefore$ $\vec{x} + \vec{y} \in S$, so S is closed under addition

(iii) S is closed under scalar multiplication.

Suppose that $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ belongs to S, so $x_1 + 2x_2 + 3x_3 = 0$.

For $t \in \mathbb{R}$, we must show that $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \end{bmatrix} \in S$.

We have

$$\begin{aligned} (tx_1) + 2(tx_2) + 3(tx_3) &= tx_1 + t(2x_2) + t(3x_3) \\ &= t(\underbrace{x_1 + 2x_2 + 3x_3}_{=0}) \\ &= t(0) \\ &= 0 \end{aligned}$$

$\therefore t\vec{x} \in S$, so S is closed under scalar multiplication.

By (i), (ii), (iii), S is a subspace of $\mathbb{R}^3$. ∎

Ex: Show that $T = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : 2x_1 = 3x_2 \right\}$ is a subspace of $\mathbb{R}^2$.

Solution: We have 3 things to check.

(i) T is non-empty

Let's check that $\vec{0} \in T$. If $x_1 = 0$ and $x_2 = 0$, then

$2x_1 = 0$ and $3x_2 = 0$, so $2x_1 = 3x_2$.

$\therefore \vec{0} \in T$, so T is non-empty.

(ii) T is closed under addition.

Suppose that $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ belong to ,

so $2x_1 = 3x_2$ and $2y_1 = 3y_2$.

We must show that $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \in T$ as well.

We have

$$2(x_1 + y_1) = \underbrace{2x_1}_{=3x_2} + \underbrace{2y_1}_{=3y_2} = 3x_2 + 3y_2 = 3(x_2 + y_2).$$

$\therefore$ $\vec{x} + \vec{y} \in T$, so T is closed under addition.

(iii) T is closed under scalar multiplication.

Suppose that $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ belongs to T, so $2x_1 = 3x_2$.

For $t \in \mathbb{R}$, we must show that $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} \in T$.

We have

$$2(tx_1) = t(\underbrace{2x_1}_{=3x_2}) = t(3x_2) = 3(tx_2).$$

$\therefore$ $t\vec{x} \in T$, so T is closed under scalar multiplication.

By (i), (ii), and (iii), T is a subspace of $\mathbb{R}^2$.

$\blacksquare$

Ex: Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 x_2 = 0 \right\}$ is not a subspace of $\mathbb{R}^2$.

Solution: The fastest way to do this would be to show that S doesn't contain $\vec{0}$. Unfortunately, $\vec{0} \in S$ ...

Can we show that S is not closed under addition?

If $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\vec{x}, \vec{y} \in S$, but

$$\vec{x} + \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S.$$

why? Because $(1)(1) \neq 0$.

So S is not closed under addition, hence S is NOT a subspace of $\mathbb{R}^2$! ∎

Exercise: For each set below, decide whether or not it is a subspace of $\mathbb{R}^3$. Justify your answer.

$$S_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 3x_2 - 5x_3 = 0 \right\}$$

$$S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 \geq 0 \right\}$$

$$S_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$S_4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + 4x_2 = 1 \right\}$$

$$S_5 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + 3x_2 = 0,\ x_2 = x_3 \right\}$$

Ex: If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, let S be the set of linear combinations of vectors from $\mathcal{B}$:

$$S = \left\{ t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k \;\middle|\; t_1, t_2, \dots, t_k \in \mathbb{R} \right\}$$

Then S is a subspace of $\mathbb{R}^n$.

Let's see why!

- S is non-empty

By taking $t_1 = t_2 = \cdots = t_k = 0$, we have that

$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k \in S.$$

$\therefore$ S is non-empty.

- S is closed under addition

Let $\vec{x}$ and $\vec{y}$ belong to S.

By definition, there are $t_1, t_2, \dots, t_k \in \mathbb{R}$ and $s_1, s_2, \dots, s_k \in \mathbb{R}$ such that

$$\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k$$

$$\vec{y} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k$$

But then

$$\vec{x} + \vec{y} = \left(t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k\right) + \left(s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k\right)$$

$$= (t_1 + s_1)\vec{v}_1 + (t_2 + s_2)\vec{v}_2 + \cdots + (t_k + s_k)\vec{v}_k$$

Since each $(t_i + s_i) \in \mathbb{R}$, we have that $\vec{x} + \vec{y} \in S$

$\therefore$ S is closed under addition

- S is closed under scalar multiplication

If $\vec{x} \in S$, then there are $s_1, s_2, \ldots, s_k \in \mathbb{R}$ such that

$$\vec{x} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k$$

Then for any $t \in \mathbb{R}$, we have

$$\begin{aligned} t\vec{x} &= t(s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k) \\ &= (ts_1)\vec{v}_1 + (ts_2)\vec{v}_2 + \cdots + (ts_k)\vec{v}_k \end{aligned}$$

Since each $ts_i \in \mathbb{R}$, we have $t\vec{x} \in S$.

$\therefore$ S is closed under scalar multiplication.

Thus, S is indeed a subspace of $\mathbb{R}^n$!

$\blacksquare$

This type of subspace shows up A LOT!

We'll study it more closely in §1.7.