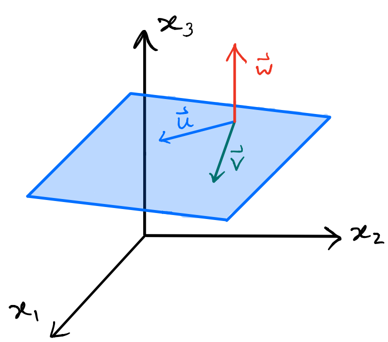
§ 1.5 - The Cross Product

We can write down the scalar equation of a plane in $\mathbb{R}^3$ if we know a non-zero vector orthogonal to the plane (i.e., a normal vector).

But how can such a vector be found??



In particular, if $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are vectors in $\mathbb{R}^3$, how can we find a vector $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ that is orthogonal to both $\vec{u}$ and $\vec{v}$?

This vector $\vec{w}$ would have to satisfy the equations

$$\begin{cases} \vec{u} \cdot \vec{w} = u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \\ \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = 0 \end{cases}$$

We'll learn how to solve equations like this in Chapter 2. For now, here's the solution:

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

or any scalar multiple of this vector.

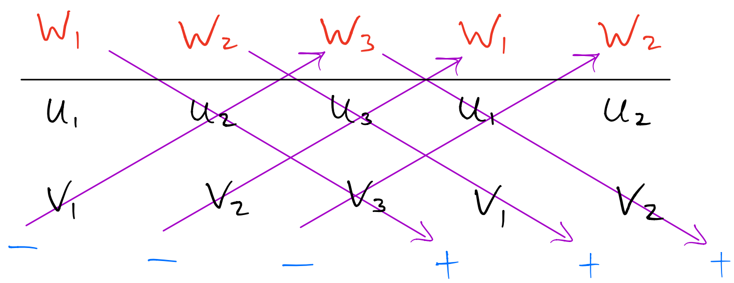
Definition: The cross product of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is the vector

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Ex: The cross product of $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ with $\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$ is

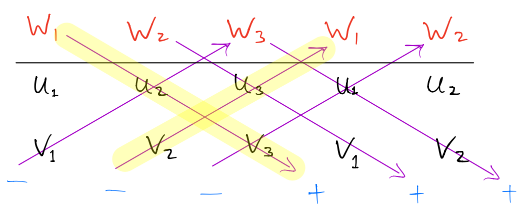
$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} (3)(7) - (4)(0) \\ (4)(1) - (2)(7) \\ (2)(0) - (3)(1) \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ -3 \end{bmatrix}$$

How can we remember this messy formula?



Suppose $\vec{u} \times \vec{v} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$.

Get w_1 by adding the product on the down arrow and subtracting the product on the up arrow.



So $w_1 = u_2 v_3 - v_2 u_3$

Do the same for w_2, w_3!

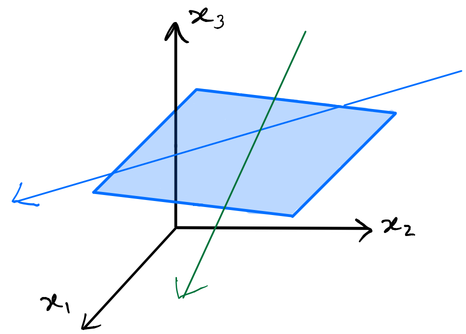
Theorem (Properties of $\times$): For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ and $t \in \mathbb{R}$,

1. $(\vec{x} \times \vec{y}) \perp \vec{x}$, $(\vec{x} \times \vec{y}) \perp \vec{y}$;
2. $\vec{x} \times \vec{y} = -(\vec{y} \times \vec{x})$;
3. $\vec{x} \times \vec{x} = \vec{0}$;
4. $\vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$;
5. $(t\vec{x}) \times \vec{y} = \vec{x} \times (t\vec{y}) = t(\vec{x} \times \vec{y})$.

Applications to Lines and Planes

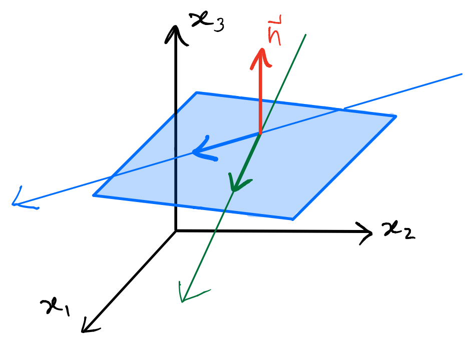
1. The equation of a plane through two lines

If two lines in $\mathbb{R}^3$ intersect at some point, then they must lie in a common plane.



What's the equation of this plane?

Idea: The plane's normal vector is orthogonal to the direction vector of each line!



Ex: The lines

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \qquad (t \in \mathbb{R})$$

both pass through $(4,0,3)$, so they must lie in a common plane. Find the scalar equation of this plane.

Solution: The normal vector of this plane is orthogonal to both direction vectors $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

So we may take

$$\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} (-1)(0) - (5)(2) \\ (5)(2) - (1)(0) \\ (1)(2) - (-1)(2) \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 4 \end{bmatrix}$$

This means that the equation is $\underline{-10x_1 + 10x_2 + 4x_3 = d}$

Solve for d by plugging in any point on the plane.
(i.e., any point on either line. $(4, 0, 3)$ will do!)

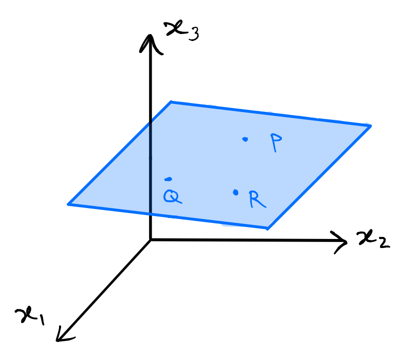
$$-10(4) + 10(0) + 4(3) = d \quad \Rightarrow \quad d = -32$$

Our plane is

$$\boxed{-10x_1 + 10x_2 + 4x_3 = -32}$$

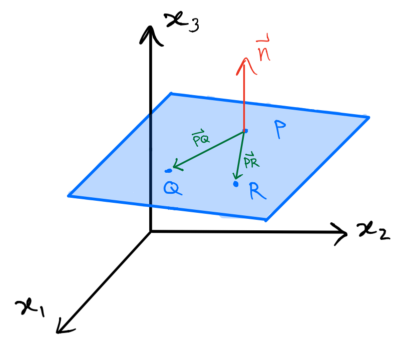
2. The equation of a plane through three points

Just as two points determine a line,
three points determine a plane.



How do we find this plane's equation?

Idea: The line segments $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in the plane, and the normal vector is orthogonal to both!



Ex: Find the scalar equation of the plane containing $P(2,-3,-1)$, $Q(5,3,5)$, and $R(0,0,2)$.

Solution: The plane contains the directed line segments

$$\overrightarrow{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix},$$

$$\overrightarrow{PR} = \vec{r} - \vec{p} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix},$$

So the plane's normal vector is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

We can take $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -21 \\ 21 \end{bmatrix}$,

$\Longrightarrow$ the equation is $\underline{-21x_2 + 21x_3 = d}$.

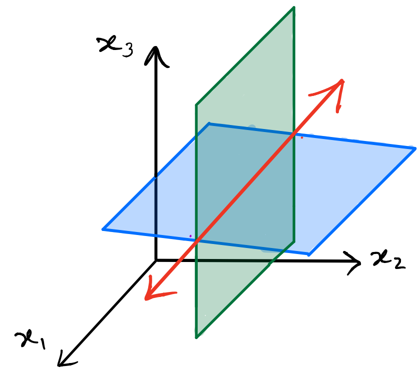
Plug in any point on the plane (e.g., P, Q, R) to get $d = 42$.

Hence, $\boxed{-21x_2 + 21x_3 = 42}$ (or $\boxed{-x_2 + x_3 = 2}$)

3. Line of intersection of two planes

Any two non-parallel planes in $\mathbb{R}^3$ intersect in a line.

If we know the planes, how can we find the line?



Idea: The line is contained in both planes, so its direction vector is orthogonal to both normal vectors!

Ex: Find the vector equation of the line in $\mathbb{R}^3$ obtained by intersecting the planes

$$x_1 - x_2 + 3x_3 = 5,$$
$$x_1 + x_2 + 2x_3 = 9.$$

Solution: The normal vectors of these planes are $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, respectively.

The direction vector of our line is orthogonal to both normals, so

$$\vec{d} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix}$$

The equation of the line is

$$\vec{x} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix} \qquad (t \in \mathbb{R})$$

where $P(p_1, p_2, p_3)$ is any point on the line.

(i.e., P is any point on both planes $x_1 - x_2 + 3x_3 = 5$, $x_1 + x_2 + 2x_3 = 9$.)

To get such a point, set $x_3 = 0$ and solve the system of equations to get x_1 and x_2.

$$\begin{cases} x_1 - x_2 + 3x_3 = 5 \\ x_1 + x_2 + 2x_3 = 9 \end{cases} \xrightarrow{\text{with } x_3 = 0} \begin{cases} x_1 - x_2 = 5 & (1) \\ x_1 + x_2 = 9 & (2) \end{cases}$$

From (1), $x_1 = x_2 + 5$

So from (2), $x_1 + x_2 = 9 \implies (x_2 + 5) + x_2 = 9$

$\implies 2x_2 = 4$

$\implies x_2 = 2$

Since $x_1 = x_2 + 5$, we get $x_1 = 7$

A point on the line is $P(7, 2, 0)$, so the equation is

$$\boxed{\vec{x} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix} \qquad (t \in \mathbb{R})}$$

Areas and Volumes

The cross product can tell us about areas in $\mathbb{R}^2$ and volumes in $\mathbb{R}^3$.

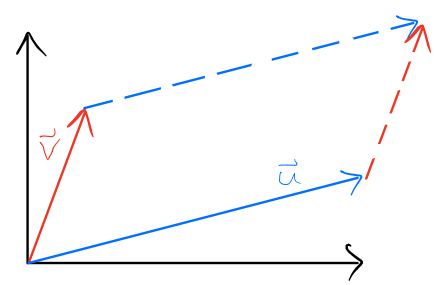
Fact: If $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^3$, and θ is the angle between them, then

$$\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta}$$

There's a proof of this fact in the book, but it's a bit messy. Instead, let's see how this formula can be used!

1. Area of Parallelogram

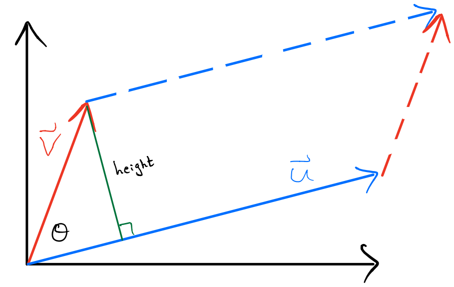
Two vectors $\vec{u}$ and $\vec{v}$ define a parallelogram



What is the area of this parallelogram?

For any parallelogram, Area = (base)(height). The base of this parallelogram has length $\|\vec{u}\|$. What's the height??

Looking at the triangle in the lower left corner, we have



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{height}}{\|\vec{v}\|}$$

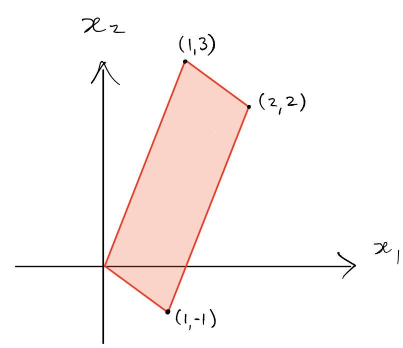
So, height $= \|\vec{v}\| \sin\theta$.

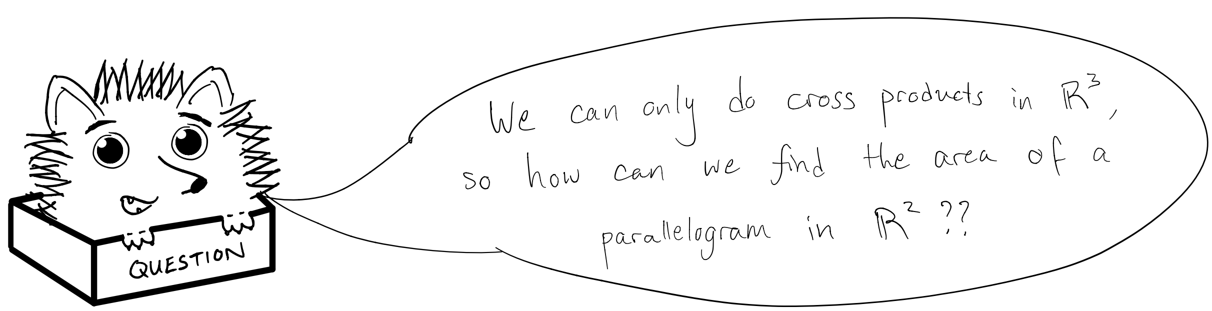
$$\Longrightarrow \quad \text{Area} = (\text{base})(\text{height}) = \|\vec{u}\| \, \|\vec{v}\| \sin\theta = \|\vec{u} \times \vec{v}\|$$

Therefore,

$$\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta = \text{Area of parallelogram defined by } \vec{u} \text{ and } \vec{v}.}$$

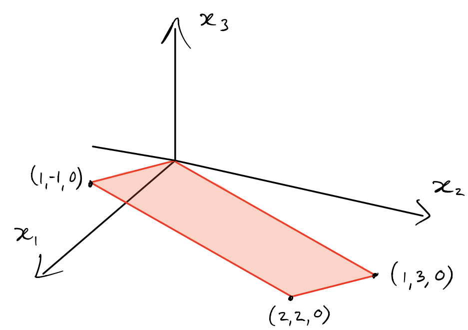
Ex: Find the area of the parallelogram on the right:





We can think of this as a parallelogram in $\mathbb{R}^3$ sitting in the x_1-x_2 plane.

(i.e., with $x_3 = 0$)



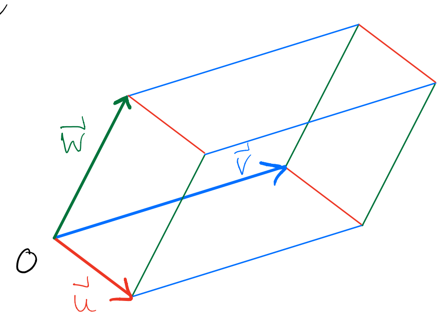
So this is the parallelogram defined by $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$.

$$\text{Area} = \|\vec{u} \times \vec{v}\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\| = \boxed{4}$$

Exercise: Calculate the area of the parallelogram defined by $\vec{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$.

2. Volume of a parallelepiped.

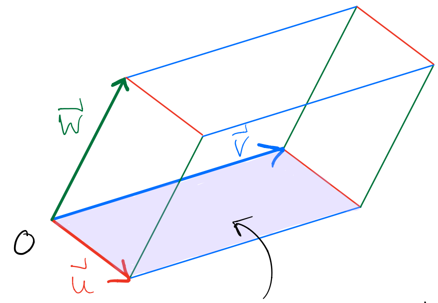
Three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ in $\mathbb{R}^3$ define a parallelepiped:



What is its volume?

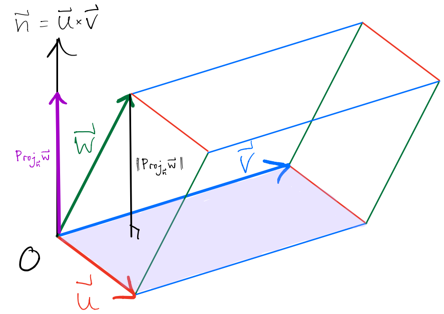
The volume of a parallelepiped is

$$\boxed{V = (\text{Area of base}) \cdot (\text{height})}$$



The area of the base is $\|\vec{u} \times \vec{v}\|$,

but what is the height?



It's exactly $\|\mathrm{proj}_{\vec{n}} \vec{w}\|$, the length of the projection of $\vec{w}$ onto $\vec{n} = \vec{u} \times \vec{v}$!

Note that

$$\|\mathrm{proj}_{\vec{n}} \vec{w}\| = \left\| \frac{\vec{w} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} \right\| = \frac{|\vec{w} \cdot \vec{n}|}{\|\vec{n}\|^{\cancel{2}}} \cancel{\|\vec{n}\|} = \frac{|\vec{w} \cdot \vec{n}|}{\|\vec{n}\|},$$

$$\begin{aligned} \text{So Volume} &= (\text{Area of base})(\text{height}) \\ &= \left(\cancel{\|\vec{u} \times \vec{v}\|}\right)\left(\frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{\cancel{\|\vec{u} \times \vec{v}\|}}\right) = \underline{|\vec{w} \cdot (\vec{u} \times \vec{v})|} \end{aligned}$$

$$\boxed{\text{The volume of the parallepiped defined by vectors } \vec{u}, \vec{v}, \vec{w} \text{ in } \mathbb{R}^3 \text{ is } |\vec{w} \cdot (\vec{u} \times \vec{v})|.}$$

Ex: What is the volume of the parallelepiped defined by $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$?

Solution: The volume is $|\vec{w} \cdot (\vec{u} \times \vec{v})|$.

We have that

$$\vec{u} \times \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

So the volume is

$$|\vec{w} \cdot (\vec{u} \times \vec{v})| = \left| \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \right| = |-3| = \boxed{3}$$

Exercise: Determine the volume of the parallelepiped defined by $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Notice something weird? Explain.

[Hint: Draw the parallelepiped!]