§1.4 — Projections

Suppose we have two vectors, $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, $\vec{x} \neq \vec{0}$.

We wish to write $\vec{y}$ as a sum of two special vectors:

$$\boxed{\vec{y} = \vec{z}_1 + \vec{z}_2}$$

where $\vec{z}_1$ is in the same direction as $\vec{x}$,

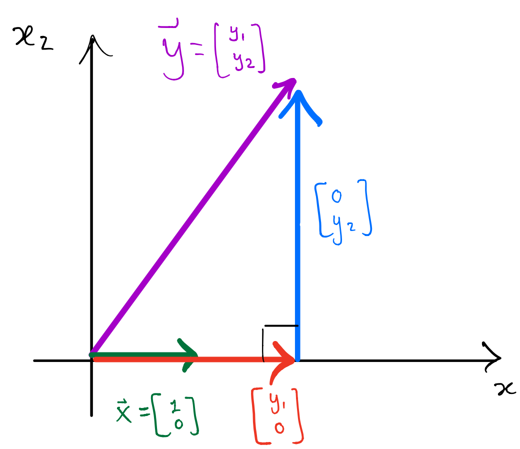
and $\vec{z}_2$ is orthogonal to $\vec{x}$.

This is easy when $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

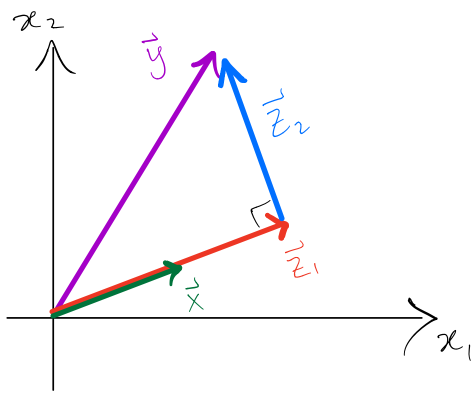
If $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then

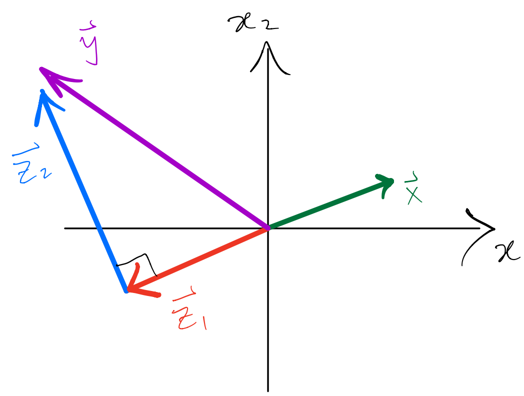
$$\vec{y} = \underbrace{\begin{bmatrix} y_1 \\ 0 \end{bmatrix}}_{\vec{z}_1} + \underbrace{\begin{bmatrix} 0 \\ y_2 \end{bmatrix}}_{\vec{z}_2}$$

(parallel to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$) (perpendicular to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$)



Q: How do we find $\vec{z}_1$, $\vec{z}_2$ when $\vec{x}$ is more complicated?





We know that $\vec{z}_1$ is parallel to $\vec{x}$ (i.e., a multiple of $\vec{x}$), so

$$\underline{\vec{z}_1 = k\vec{x} \quad \text{for some } k \in \mathbb{R}.}$$

This means that

$$\underline{\vec{y} = \vec{z}_1 + \vec{z}_2 = k\vec{x} + \vec{z}_2}$$

where $\vec{z}_2$ is orthogonal to $\vec{x}$.

To find k, take the dot product of $\vec{x}$ and $\vec{y}$:

$$\vec{x}\cdot\vec{y} = \vec{x}\cdot(k\vec{x} + \vec{z}_2) = k\underbrace{(\vec{x}\cdot\vec{x})}_{=\|\vec{x}\|^2} + \underbrace{(\vec{x}\cdot\vec{z}_2)}_{=0} \quad \text{as } \vec{z}_2 \perp \vec{x}\,!$$

So... $\quad \vec{x}\cdot\vec{y} = K\|\vec{x}\|^2 \implies K = \dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}$

We've just shown that $\boxed{\vec{z}_1 = \left(\dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\right)\vec{x}\ !}$

Since $\vec{y} = \vec{z}_1 + \vec{z}_2$, $\boxed{\vec{z}_2 = \vec{y} - \vec{z}_1}$

Definition: If $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$ with $\vec{x} \neq \vec{0}$, then

the projection of $\vec{y}$ onto $\vec{x}$ is $\boxed{\operatorname{proj}_{\vec{x}}\vec{y} = \left(\dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\right)\vec{x}}$

and the perpendicular part is $\boxed{\operatorname{perp}_{\vec{x}}\vec{y} = \vec{y} - \operatorname{proj}_{\vec{x}}\vec{y}}$.

From this definition it's clear that

(i) $\vec{y} = \operatorname{proj}_{\vec{x}}\vec{y} + \operatorname{perp}_{\vec{x}}\vec{y}$, and

(ii) $\operatorname{proj}_{\vec{x}}\vec{y}$ is a multiple of $\vec{x}$.

In Q2 of Assignment 2, you verify

(iii) $\operatorname{perp}_{\vec{x}}\vec{y} = \vec{y} - \left(\dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\right)\vec{x}$ is orthogonal to $\vec{x}$.

Ex: What is the projection of $\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ onto $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

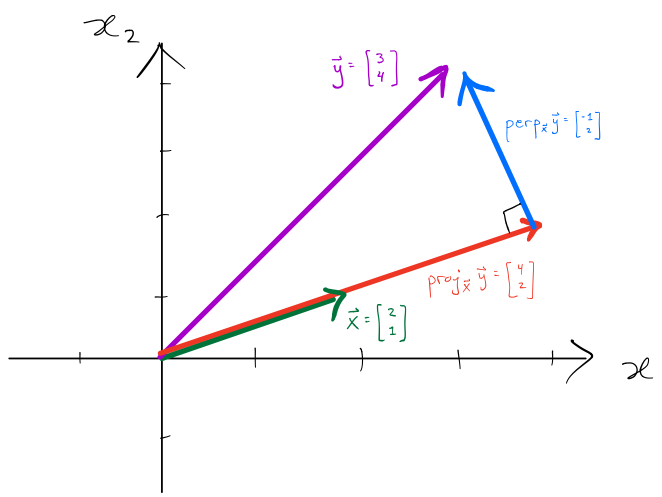
What is the perpendicular part?

Solution: We have that $\text{proj}_{\vec{x}} \vec{y} = \left(\dfrac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2} \right) \vec{x}$.

We get $\|\vec{x}\|^2 = \left(\sqrt{2^2 + 1^2} \right)^2 = 5$ and $\vec{x} \cdot \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 10$

So $\text{proj}_{\vec{x}} \vec{y} = \left(\dfrac{10}{5} \right) \vec{x} = 2\vec{x} = \boxed{\begin{bmatrix} 4 \\ 2 \end{bmatrix}}$

and $\text{perp}_{\vec{x}} \vec{y} = \vec{y} - \text{proj}_{\vec{x}} \vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$



Ex: If $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ -10 \\ 0 \end{bmatrix}$, find $\text{proj}_{\vec{x}} \vec{y}$, $\text{perp}_{\vec{x}} \vec{y}$,

and $\text{proj}_{\vec{y}} \vec{x}$.

Solution: Note that $\|\vec{x}\|^2 = 10$ and $\vec{x}\cdot\vec{y} = -10$

So $\text{proj}_{\vec{x}}\vec{y} = \left(\frac{\vec{x}\cdot\vec{y}}{\|x\|^2}\right)\vec{x} = \left(\frac{-10}{10}\right)\vec{x} = -\vec{x} = \boxed{\begin{bmatrix}0\\-1\\-3\end{bmatrix}}$

$$\text{perp}_{\vec{x}}\vec{y} = \vec{y} - \text{proj}_{\vec{x}}\vec{y} = \begin{bmatrix}2\\-10\\0\end{bmatrix} - \begin{bmatrix}0\\-1\\-3\end{bmatrix} = \boxed{\begin{bmatrix}2\\-9\\3\end{bmatrix}}$$

For $\text{proj}_{\vec{y}}\vec{x}$, note that $\|\vec{y}\|^2 = 104$ and $\vec{y}\cdot\vec{x} = -10$.

So $\text{proj}_{\vec{y}}\vec{x} = \left(\frac{\vec{y}\cdot\vec{x}}{\|\vec{y}\|^2}\right)\vec{y} = \left(\frac{-10}{104}\right)\vec{y} = \boxed{\begin{bmatrix}-20/104\\100/104\\0\end{bmatrix}}$

Remark: The above example shows that, in general,

$$\text{proj}_{\vec{x}}\vec{y} \neq \text{proj}_{\vec{y}}\vec{x}.$$

That's not too surprising; $\text{proj}_{\vec{x}}\vec{y}$ is a multiple of $\vec{x}$, while $\text{proj}_{\vec{y}}\vec{x}$ is a multiple of $\vec{y}$!

Theorem (Properties of Proj/Perp)

Let $\vec{x}, \vec{y}, \vec{z}$ be vectors in $\mathbb{R}^n$, and let $t \in \mathbb{R}$

1. $\text{Proj}_{\vec{x}}(\vec{y}+\vec{z}) = \text{Proj}_{\vec{x}}\vec{y} + \text{Proj}_{\vec{x}}\vec{z}$

2. $\text{Proj}_{\vec{x}}(t\vec{y}) = t\,\text{Proj}_{\vec{x}}\vec{y}$

3. $\text{Proj}_{\vec{x}}(\text{Proj}_{\vec{x}}\vec{y}) = \text{Proj}_{\vec{x}}(\vec{y})$

These properties are also true for $\text{Perp}_{\vec{x}}$.

Properties 1. and 2. say that $\text{proj}_{\vec{x}}$ and $\text{perp}_{\vec{x}}$ are linear functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ (something we'll return to in Chapter 3).

Proof of 1.:

$$\text{Proj}_{\vec{x}}(\vec{y}+\vec{z}) = \left(\frac{\vec{x}\cdot(\vec{y}+\vec{z})}{\|\vec{x}\|^2}\right)\vec{x}$$

$$= \left(\frac{\vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}}{\|\vec{x}\|^2}\right)\vec{x}$$

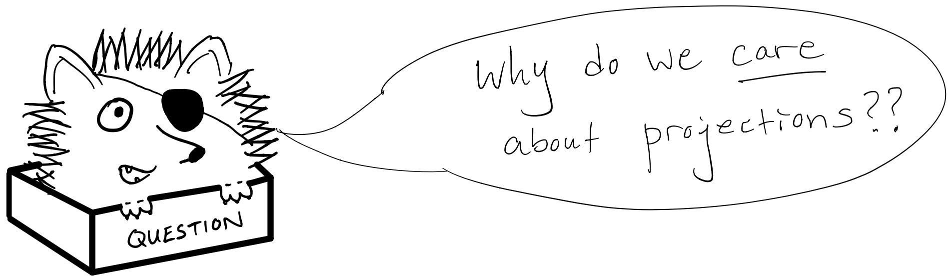
$$= \left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2} + \frac{\vec{x}\cdot\vec{z}}{\|\vec{x}\|^2}\right)\vec{x}$$

$$= \left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\right)\vec{x} + \left(\frac{\vec{x}\cdot\vec{z}}{\|\vec{x}\|^2}\right)\vec{x} = \text{Proj}_{\vec{x}}\vec{y} + \text{Proj}_{\vec{x}}\vec{z}. \quad \blacksquare$$

Exercise: Try a similar approach for 2.

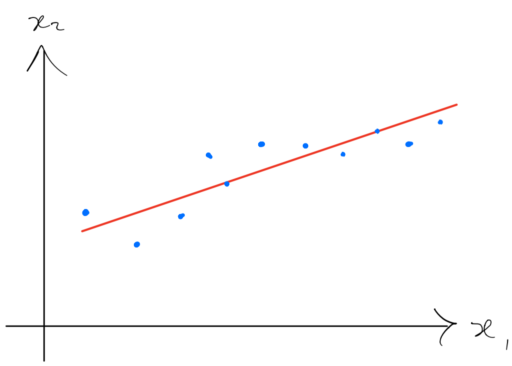
Can you explain intuitively why 3. is true?

Application: Minimum Distance



There are lots of reasons to care about projections.

One major application: projections can be used to find the line of best fit for a set of data points.



This is the line whose total distance to a set of points is as small as possible.†

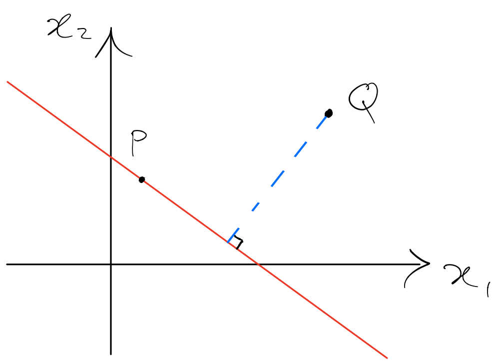
A complete treatment of this topic: MATH 235.

Instead, we'll look at two different applications of projections to minimum distance.

† Want to know more about how to find this line? Talk to me after Chapter 3!

1. Distance from Point to Line.

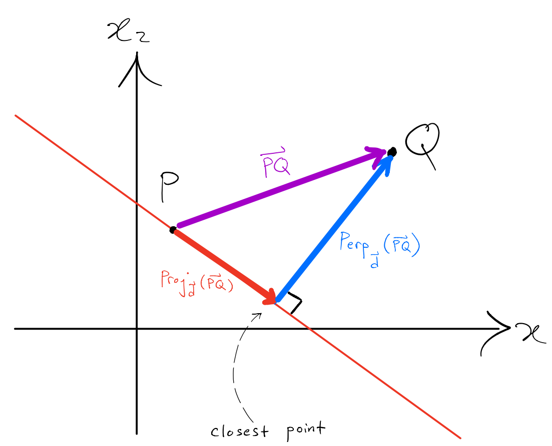
Q: What's the distance (i.e., shortest distance) from a point Q to a line $\vec{x} = \vec{p} + t\vec{d}$ $(t \in \mathbb{R})$?



It's the length of the dotted line that meets our line at a right angle.

In the language of projections, this is exactly $\boxed{\| \mathrm{Perp}_{\vec{d}}(\vec{PQ}) \|}$

The point on the line closest to Q is $\boxed{\vec{p} + \mathrm{Proj}_{\vec{d}}(\vec{PQ})}$



Ex: Find the distance from $Q(0,2)$ to the line

$\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. What is the closest point?

Solution: We have that $\vec{p} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

From the above, the distance $= \|\operatorname{perp}_{\vec{d}}(\vec{PQ})\|$

and the closest point is $\vec{p} + \operatorname{Proj}_{\vec{d}}(\vec{PQ})$.

So... let's find $\operatorname{Proj}_{\vec{d}}(\vec{PQ})$ and $\operatorname{Perp}_{\vec{d}}(\vec{PQ})$!

Note: $\vec{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$,

$\|\vec{d}\|^2 = 2$,

$\vec{d} \cdot \vec{PQ} = -4$.

So $\operatorname{Proj}_{\vec{d}} \vec{PQ} = \left(\frac{\vec{d} \cdot \vec{PQ}}{\|\vec{d}\|^2}\right)\vec{d} = \left(\frac{-4}{2}\right)\vec{d} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$,

$\operatorname{Perp}_{\vec{d}} \vec{PQ} = \vec{PQ} - \operatorname{Proj}_{\vec{d}} \vec{PQ} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

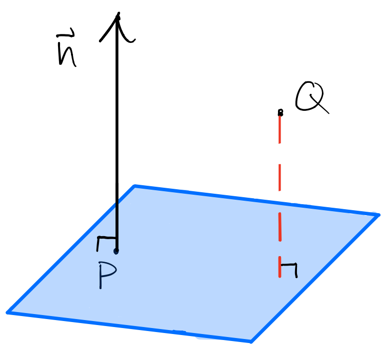
The distance is $\|\operatorname{Perp}_{\vec{d}}(\vec{PQ})\| = \left\|\begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\| = \boxed{\sqrt{8}}$

The closest point is $\vec{p} + \operatorname{Proj}_{\vec{d}}(\vec{PQ}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$

Exercise: Find the distance from $Q(1,0,1)$ to the line $\vec{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. What is the closest point?

2. Distance from Point to Plane

Q: What's the distance from a point Q to a plane in $\mathbb{R}^3$ with normal vector $\vec{n}$?

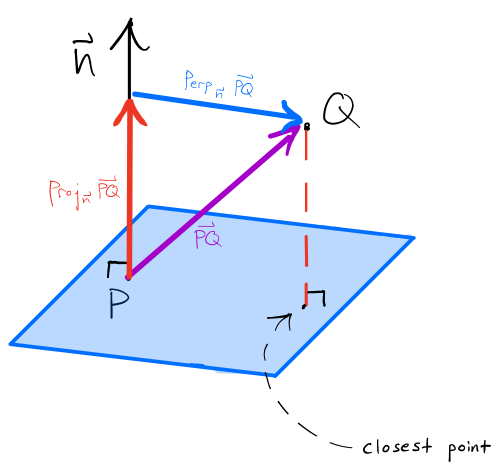


Suppose P is any point on the plane.

The distance is the length of the dotted line segment that is orthogonal to the plane.

In terms of projections, this is $\boxed{\left\| \operatorname{Proj}_{\vec{n}} (\vec{PQ}) \right\|}$

The point on the plane closest to Q is $\boxed{\vec{q} - \operatorname{Proj}_{\vec{n}} \vec{PQ}.}$



(This is the same as $\vec{p} + \operatorname{Perp}_{\vec{d}} \vec{PQ}$)

Ex: What is the distance from $Q(-1,1,2)$ to the plane $x_1 + 2x_2 + 2x_3 = -4$? Find the point on the plane that is closest to Q.

Solution: We can get a point P on the plane by setting $x_2 = x_3 = 0$ and solving for x_1

$$x_1 + 2x_2 + 2x_3 = -4 \quad \xrightarrow[x_2 = x_3 = 0]{} \quad x_1 = -4$$

So $P = (-4, 0, 0)$ is a point on the plane, and $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

From above, we know that the distance is $\|\text{Proj}_{\vec{n}} \overrightarrow{PQ}\|$ and the closest point is $\underset{\sim}{q} - \text{Proj}_{\vec{n}} \overrightarrow{PQ}$.

We have $\overrightarrow{PQ} = \underset{\sim}{q} - \vec{p} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$,

$\|\vec{n}\|^2 = 9$,

$\vec{n} \cdot \overrightarrow{PQ} = 9$.

So $$\text{Proj}_{\vec{n}} \overrightarrow{PQ} = \left(\frac{\vec{n} \cdot \overrightarrow{PQ}}{\|\vec{n}\|^2} \right) \vec{n} = \left(\frac{\cancel{9}}{\cancel{9}} \right) \vec{n} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

The distance is $\|\text{Proj}_{\vec{n}} \overrightarrow{PQ}\| = \sqrt{9} = \boxed{3}$

The closest point is

$$\vec{q} - \operatorname{Proj}_{\vec{n}} \overrightarrow{PQ} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}}$$

Exercise: What is the distance from $Q(1,1,1)$ to the plane $2x_1 - x_2 + x_3 = 2$?

Notice anything odd?

Which point on this plane is closest to Q?