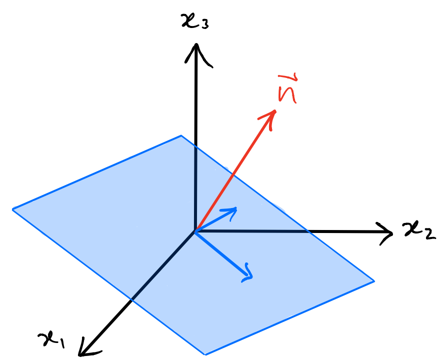
§ 1.3 – Planes & Hyperplanes

Let $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ be a non-zero vector in $\mathbb{R}^3$.

Consider all vectors $\vec{x}$ that are orthogonal to $\vec{n}$.

These vectors form a plane passing through the origin.



The scalar equation of a plane through the origin that is orthogonal to $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is $\vec{n} \cdot \vec{x} = 0$, or

$$\boxed{n_1 x_1 + n_2 x_2 + n_3 x_3 = 0.}$$

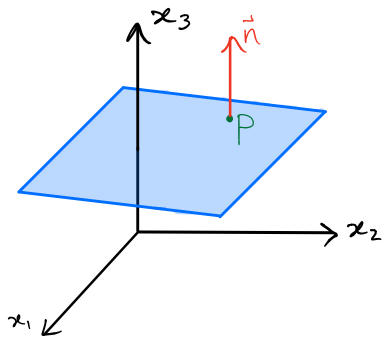
A non-zero vector that is orthogonal to the plane (e.g., $\vec{n}$) is called a normal vector of the plane.

Ex: Find the scalar equation of a plane through the origin in $\mathbb{R}^3$ that has $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ as a normal vector.

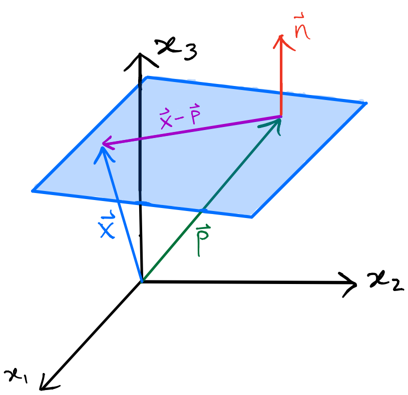
Solution: The plane contains all vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ orthogonal to $\vec{n}$ (i.e., $\vec{n} \cdot \vec{x} = 0$)

Thus, the equation is $\boxed{2x_1 + 3x_2 - x_3 = 0.}$

What if the plane passes through a point $P(p_1, p_2, p_3)$, but not necessarily through the origin?



Suppose $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is a normal vector.



If $X(x_1, x_2, x_3)$ is on this plane, then $\overline{PX} = \vec{x} - \vec{p}$ is a line segment in the plane.

Hence, $\vec{x} - \vec{p}$ is orthogonal to $\vec{n}$.

So the equation of our plane is $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$, or

$$\boxed{n_1(x_1 - p_1) + n_2(x_2 - p_2) + n_3(x_3 - p_3) = 0}$$

This can also be written as

$$\boxed{n_1x_1 + n_2x_2 + n_3x_3 = d}$$

where

$$\boxed{d = n_1p_1 + n_2p_2 + n_3p_3 = \vec{n}\cdot\vec{p}.}$$

Ex: Find the scalar equation of the plane in $\mathbb{R}^3$ that passes through $(0,-2,3)$ and is orthogonal to $\begin{bmatrix}4\\1\\-1\end{bmatrix}$.

Solution: The normal vector is $\vec{n} = \begin{bmatrix}4\\-1\\1\end{bmatrix}$, and a point on the plane is $P(0,-2,3)$.

The equation is $\underline{n_1(x_1-p_1) + n_2(x_2-p_2) + n_3(x_3-p_3) = 0}$

So $4(x_1-0) + (-1)(x_2-(-2)) + (x_3-3) = 0$

$$\Rightarrow \boxed{4x_1 - (x_2+2) + (x_3-3) = 0}$$

OR $\boxed{4x_1 - x_2 + x_3 = 5}$ $\quad (\underline{n_1x_1 + n_2x_2 + n_3x_3 = d})$

Definition: Two planes in $\mathbb{R}^3$ are parallel if the normal vector of one plane is a non-zero scalar multiple of the normal vector of the other.

Two planes are orthogonal (perpendicular) if their normal vectors are orthogonal.

Ex: The planes
$$2x_1 + x_2 + 3x_3 = 0$$
$$4x_1 + 2x_2 + 6x_3 = 12$$
are parallel, as their normal vectors are $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, respectively.

Exercise: Are they parallel to $-x_1 - \frac{1}{2}x_2 - \frac{3}{2}x_3 = 7$?

What about to $x_1 + x_2 - x_3 = -1$?

Ex: The planes
$$4x_1 + 2x_2 + 6x_3 = 12$$
$$x_1 + x_2 - x_3 = -1$$
are orthogonal, as their normal vectors are $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and

$$\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = (4)(1) + (2)(1) + (6)(-1) = 0.$$

Exercise: Is either plane orthogonal to
$$2x_1 + x_2 + 3x_3 = 0 \text{ ?}$$

Ex: Find a scalar equation of the plane in $\mathbb{R}^3$ that passes through $(1,2,3)$ and is parallel to $4x_1 - x_2 + 10x_3 = 7$

Solution: The normal vector for the parallel plane is $\begin{bmatrix} 4 \\ -1 \\ 10 \end{bmatrix}$, so our normal vector $\vec{n}$ can be any (non-zero) multiple of this vector.

Let's take $\vec{n} = \begin{bmatrix} 4 \\ -1 \\ 10 \end{bmatrix}$, and $\vec{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is given.

The plane is $\boxed{4(x_1 - 1) - 1(x_2 - 2) + 10(x_3 - 3) = 0}$

OR $\boxed{4x_1 - x_2 + 10x_3 = -32}$

Ex: Find a scalar equation of the plane in $\mathbb{R}^3$ that passes through $(1,2,3)$ and is orthogonal to $4x_1 - x_2 + 10x_3 = 7$

Solution: Our normal vector must be orthogonal to $\begin{bmatrix} 4 \\ -1 \\ 10 \end{bmatrix}$.

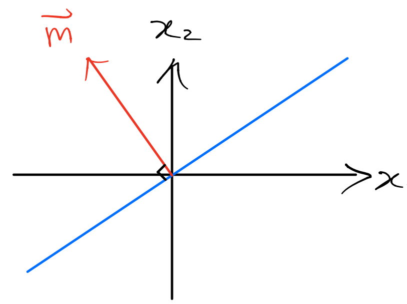
I think $\begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix}$ works. Can you think of another example?

Try to get the equation of the plane from here!

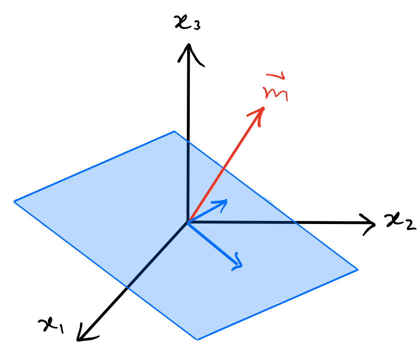
Hyperplanes:

A hyperplane through the origin in $\mathbb{R}^n$ is a collection of vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ that are all orthogonal to a given vector $\vec{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}$.

In $\mathbb{R}^2$, this is a line through the origin.



In $\mathbb{R}^3$, this is a plane through the origin



The equation of a hyperplane through the origin is

$$\vec{m} \cdot \vec{x} = 0,$$

OR

$$\boxed{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n = 0.}$$

We can also talk about hyperplanes that pass through a point $P(p_1, p_2, \ldots, p_n)$ but not necessarily through the origin.

The same work we did for planes in $\mathbb{R}^3$ shows that a hyperplane through $P(p_1, p_2, \ldots, p_n)$ and orthogonal to $\vec{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}$ has scalar equation

$$\vec{m} \cdot (\vec{x} - \vec{p}) = 0.$$

That is, $\boxed{m_1(x_1 - p_1) + m_2(x_2 - p_2) + \cdots + m_n(x_n - p_n) = 0}$,

which can also be written as

$$\boxed{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n = \vec{m} \cdot \vec{p}}$$

Ex: Find the scalar equation of the hyperplane in $\mathbb{R}^4$ that has normal vector $\begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}$ and passes through $(4,1,8,2)$.

Solution: Here, $\vec{m} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}$ and $\vec{p} = \begin{bmatrix} 4 \\ 1 \\ 8 \\ 2 \end{bmatrix}$ so the equation is

$$1(x_1 - 4) + 0(x_2 - 1) + 2(x_3 - 8) - 3(x_4 - 2) = 0$$

$$\Rightarrow \boxed{(x_1 - 4) + 2(x_3 - 8) - 3(x_4 - 2) = 0}$$

OR

$$\boxed{x_1 + 2x_2 - 3x_4 = 14}$$