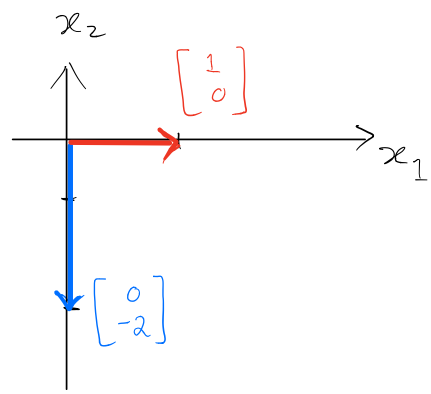
§ 1.2 – Length; Dot Product; Orthogonality

Length

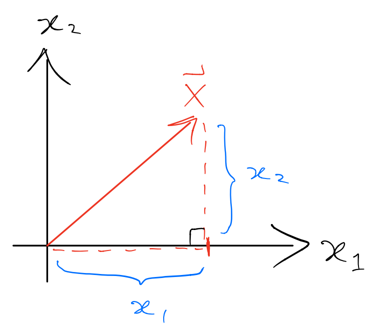
The length (or norm) of a vector $\vec{x} \in \mathbb{R}^n$ is denoted by $\|\vec{x}\|$.

Ex:
$$\left\|\begin{bmatrix}1\\0\end{bmatrix}\right\| = 1$$
$$\left\|\begin{bmatrix}0\\-2\end{bmatrix}\right\| = 2$$
$$\left\|\begin{bmatrix}0\\0\end{bmatrix}\right\| = 0$$



If $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ in $\mathbb{R}^2$,

what is $\|\vec{x}\|$?



By the Pythagorean Theorem, we have

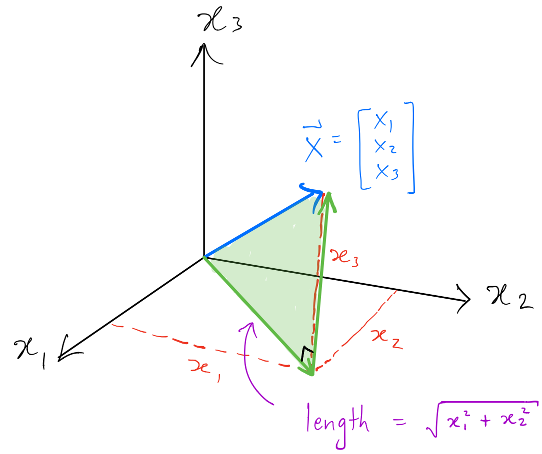
$$\|\vec{x}\|^2 = x_1^2 + x_2^2$$

So
$$\boxed{\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}}$$

Ex: $\left\| \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\| = \sqrt{(-3)^2 + 5^2} = \sqrt{9+25} = \boxed{\sqrt{34}}$

What about $\left\| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\|$ in $\mathbb{R}^3$?

$\vec{x}$ is the sum of the two vectors in the green right triangle.



So, $\|\vec{x}\|^2 = \left(\sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2$ (Pythagoras!)

$= (x_1^2 + x_2^2) + x_3^2$

$\Rightarrow \boxed{\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}\,.}$

Notice a pattern?

Definition: The length (or norm) of a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ is

$$\boxed{\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$$

If $\|\vec{x}\| = 1$, $\vec{x}$ is called a unit vector.

Ex: If $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} \in \mathbb{R}^4$, then

$$\|\vec{x}\| = \sqrt{3^2 + 1^2 + 0^2 + (-1)^2} = \sqrt{9+1+0+1} = \boxed{\sqrt{11}}$$

Theorem (Properties of $\|\cdot\|$): Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, and $t \in \mathbb{R}$.

1. $\|\vec{x}\| \geqslant 0$, and $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$.
2. $\|t\vec{x}\| = |t| \, \|\vec{x}\|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. (Triangle Inequality)

Think about 1. Why is $\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ always $\geqslant 0$?

When is it $= 0$?

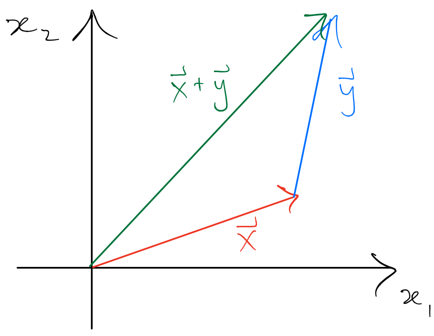
For 2., we have

$$\|t\vec{x}\| = \sqrt{(tx_1)^2 + (tx_2)^2 + \dots + (tx_n)^2}$$

$$= \sqrt{t^2 x_1^2 + t^2 x_2^2 + \dots + t^2 x_n^2}$$

$$= \sqrt{t^2}\sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = |t| \, \|\vec{x}\|.$$

Geometrically, 3. says that the shortest path between two points is a straight line.



$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

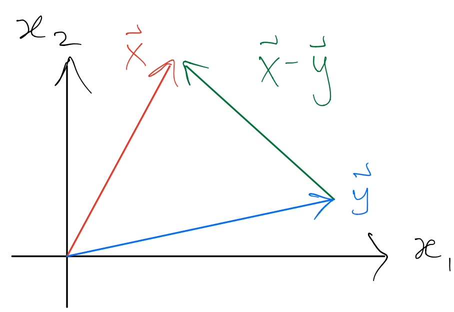
Exercise: What is the norm of $\frac{1}{\|\vec{x}\|}\vec{x}$ if

(a) $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}$?

(b) $\vec{x}$ is any non-zero vector in $\mathbb{R}^n$?

The notion of length also allows us to measure distance!

The distance from $\vec{x}$ to $\vec{y}$ in $\mathbb{R}^n$ is $\|\vec{x}-\vec{y}\|$.

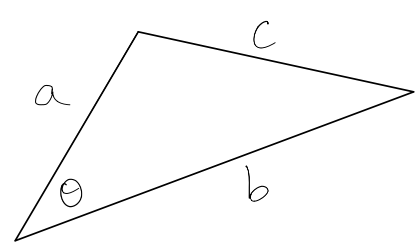


Exercise: Compute the distance from $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ to $\vec{y} = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$.

Angles & Dot Product

Enough with vectors! Let's go back to highschool.

Q: If a, b, c are given, what is the value of θ?



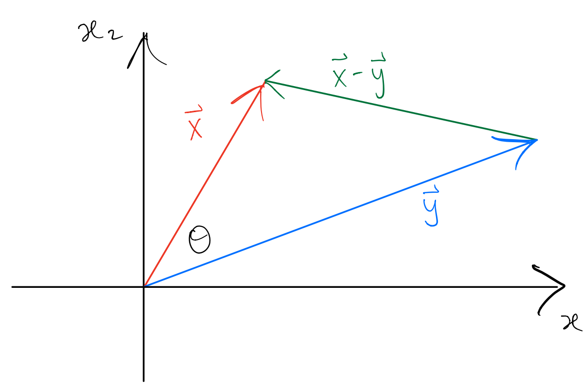
The cosine law can help us:

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$$

New Q: If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, what is the angle θ between $\vec{x}$ and $\vec{y}$?

Compare the picture on the right with the one above.

What does the cosine law say now??



$$\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\Rightarrow (x_1-y_1)^2+(x_2-y_2)^2 = (x_1^2+x_2^2)+(y_1^2+y_2^2)-2\|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\Rightarrow (\cancel{x_1^2}-2x_1y_1+\cancel{y_1^2})+(\cancel{x_2^2}-2x_2y_2+\cancel{y_2^2})$$

$$= (\cancel{x_1^2}+\cancel{x_2^2})+(\cancel{y_1^2}+\cancel{y_2^2})-2\|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\Rightarrow -\cancel{2}(x_1y_1+x_2y_2) = -\cancel{2}\|\vec{x}\|\|\vec{y}\|\cos\theta$$

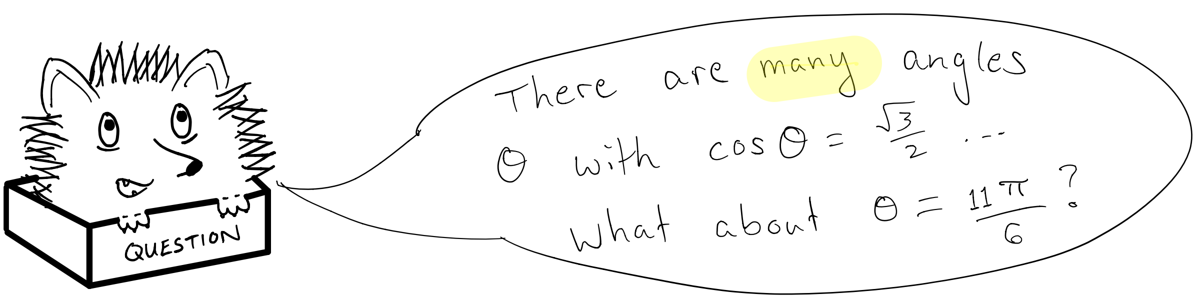
$$\Rightarrow \boxed{x_1y_1+x_2y_2 = \|\vec{x}\|\|\vec{y}\|\cos\theta}$$

$\blacksquare$ (end of proof)

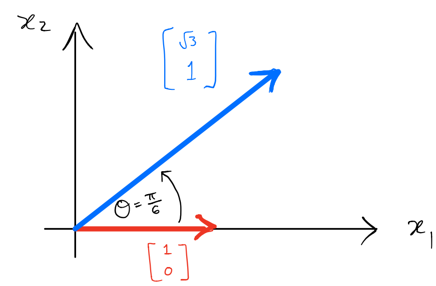
Ex: If $\vec{x}=\begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{y}=\begin{bmatrix}\sqrt{3}\\1\end{bmatrix}$, then

$\|\vec{x}\|=1$, $\|\vec{y}\|=2$, and $x_1y_1+x_2y_2=\sqrt{3}$.

So $\sqrt{3}=1\cdot 2\cdot\cos\theta \Rightarrow \cos\theta=\frac{\sqrt{3}}{2} \Rightarrow \boxed{\theta=\pi/6}$



Note: We always pick θ so that $0 \le \theta \le \pi$.



Something similar happens in $\mathbb{R}^3$:

If θ is the angle between $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then

$$\boxed{x_1 y_1 + x_2 y_2 + x_3 y_3 = \|\vec{x}\| \|\vec{y}\| \cos\theta}$$

Definition: Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$.

The dot product of $\vec{x}$ and $\vec{y}$ is

$$\boxed{\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n}$$

Ex: $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = (3)(-4) + (2)(0) + (-1)(1)$

$= (-12) + 0 + (-1)$

$= \boxed{-13}$

Theorem (Properties of $\cdot$): Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, and $t \in \mathbb{R}$.

1. $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2 \longleftarrow$ Super useful!!
2. $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
3. $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$
4. $(t\vec{x}) \cdot \vec{y} = t(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (t\vec{y})$
5. $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$ (Cauchy-Schwarz Inequality)

Try to prove 1. - 4. on your own.

The book has a proof of 5. (harder)

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the angle between $\vec{x}$ and $\vec{y}$ is the $\theta \in [0, \pi]$ such that

$$\boxed{\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta.}$$

We can use this formula to define the angle between (non-zero) vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$!

It's the $\theta \in [0, \pi]$ such that $\boxed{\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta.}$

(The Cauchy-Schwarz Inequality says θ always exists!)

Exercise: Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) Compute $\|\vec{x}\|$, $\|\vec{y}\|$, and $\vec{x} \cdot \vec{y}$.

(b) Use the above formula to show that $\vec{x}$ and $\vec{y}$ meet at an angle of ≈ 1.83 radians.

Orthogonality:

We know that dot products are related to angles by the cosine formula above.

Q: What does it mean if $\vec{x} \cdot \vec{y} = 0$ for two non-zero vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$?

In this case $0 = \underbrace{\|\vec{x}\|}_{\neq 0} \underbrace{\|\vec{y}\|}_{\neq 0} \cos\theta$, so $\cos\theta = 0$.

Then $\theta = \pi/2$, so $\vec{x}$ and $\vec{y}$ are perpendicular!

Conversely, if $\vec{x}$ and $\vec{y}$ are perpendicular, then they meet at an angle of $\theta = \pi/2$. Hence,

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \underbrace{\cos(\pi/2)}_{=0} = 0$$

We can therefore make the following definition:

Definition: Vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are perpendicular (or orthogonal) if and only if $\vec{x} \cdot \vec{y} = 0$.

When $\vec{x}$ is orthogonal to $\vec{y}$, we write $\vec{x} \perp \vec{y}$

Ex: $\vec{0}$ is orthogonal to all $\vec{x} \in \mathbb{R}^n$, as $\vec{0} \cdot \vec{x} = 0$.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$, So these vectors are orthogonal.

$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1$, So these vectors are NOT orthogonal.

Two lines in $\mathbb{R}^n$ are orthogonal if their direction vectors are orthogonal.

Exercise: Which of the following pairs of lines are orthogonal?

(i) $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix}$; $\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + t\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ $(t \in \mathbb{R})$

(ii) $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 6 \\ 1 \\ -1 \\ -2 \end{bmatrix}$; $\vec{x} = t\begin{bmatrix} -1 \\ 2 \\ 2 \\ -3 \end{bmatrix}$ $(t \in \mathbb{R})$

(iii) $\begin{cases} x_1 = 2 + t \\ x_2 = 4 - 2t \\ x_3 = 1 - 5t \end{cases}$; $\begin{cases} x_1 = -9 + 6t \\ x_2 = 3 + 3t \\ x_3 = 2 \end{cases}$ $(t \in \mathbb{R})$

(iv) The line passing through $P(2,3)$ and $Q(4,5)$; The line passing through $R(1,2)$ and parallel to $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $(t \in \mathbb{R})$