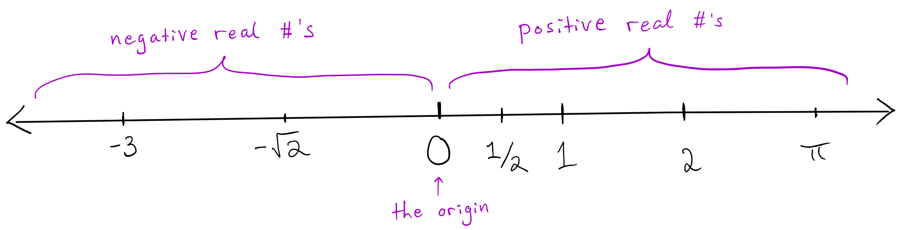
Chapter 1: Vector Geometry

§1.1 – Vectors & Lines

$\mathbb{R}$ = the set of all real numbers.

e.g.: $1 \overset{\text{"belongs to"}}{\in} \mathbb{R}$, $2 \in \mathbb{R}$, $-3 \in \mathbb{R}$, $\pi \in \mathbb{R}$, $\frac{1}{2} \in \mathbb{R}$, $-\sqrt{2} \in \mathbb{R}$

We often think of $\mathbb{R}$ as a 1-dimensional space (the real line):



$\mathbb{R}$ has many nice properties... here are two very special ones:

① We can add!

$$\boxed{\text{If } x \in \mathbb{R} \text{ and } y \in \mathbb{R}, \text{ then } x+y \in \mathbb{R}.}$$

($\mathbb{R}$ is closed under addition)

② We can scale/stretch!

$$\boxed{\text{If } x \in \mathbb{R}, \text{ then } tx \in \mathbb{R} \text{ for all real numbers } t.}$$

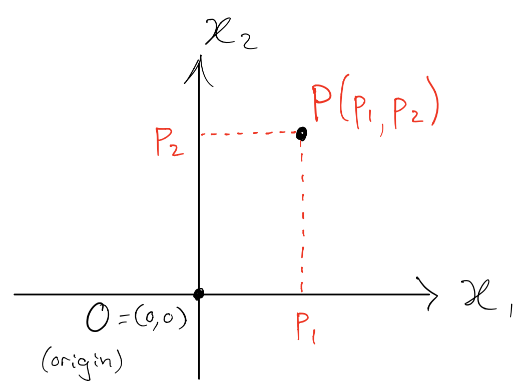
($\mathbb{R}$ is closed under scalar multiplication)

Loosely speaking, a set with properties 1 + 2 is called a real vector space or a vector space over $\mathbb{R}$.

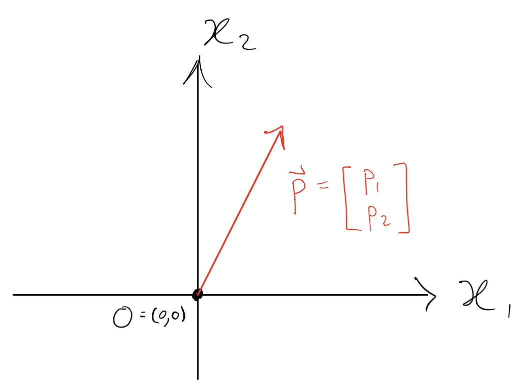
In this course, we'll study real vector spaces in higher dimensions.

$\mathbb{R}^2$: 2-Dimensional Space

We can represent elements in 2-dimensions as points...



BUT it can also be very useful to view them as arrows starting at the origin.



These arrows are called vectors.

Notation: The point $X(x_1, x_2)$ corresponds to the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

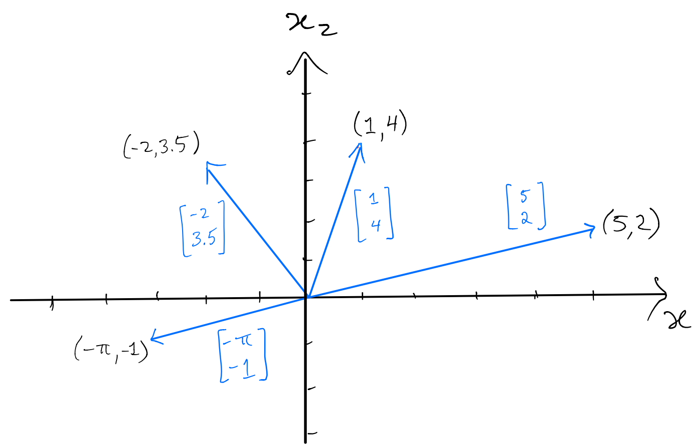
The zero vector is $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Ex: The vectors

$\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix}, \begin{bmatrix}-2\\3.5\end{bmatrix}, \begin{bmatrix}-\pi\\-1\end{bmatrix}$ and

their corresponding points

are plotted on the right.



Definition: $\mathbb{R}^2$ is the set of all vectors of the form

$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$, where x_1, x_2 are real numbers, i.e.,

$$\boxed{\mathbb{R}^2 = \left\{ \begin{bmatrix}x_1\\x_2\end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}}$$

We'll often switch between thinking of elements in $\mathbb{R}^2$

as points $P(p_1, p_2)$ and vectors $\vec{p} = \begin{bmatrix}p_1\\p_2\end{bmatrix}$

Q: Can we add and scale/stretch in $\mathbb{R}^2$? <u>YES!</u>

Addition: If $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$, then

$$\boxed{\vec{x} + \vec{y} = \begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}}$$

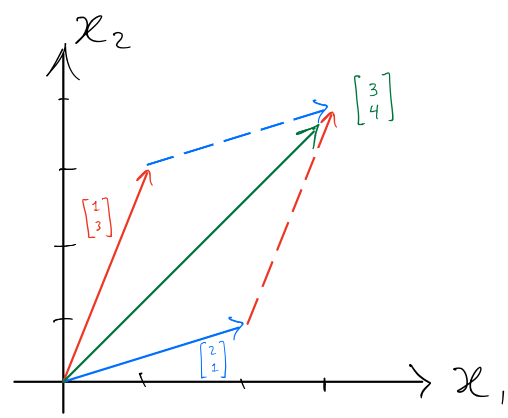
Scalar Multiplication: If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $t \in \mathbb{R}$, then

$$\boxed{t\vec{x} = t\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix}}$$

These operations can be viewed geometrically!

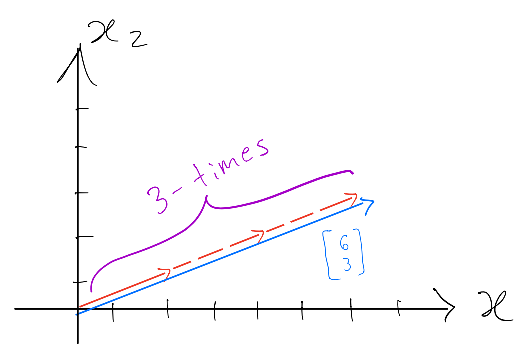
To add, connect vectors tip to tail.

Ex: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



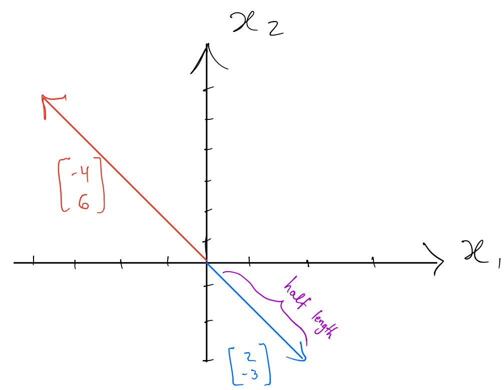
To scalar multiply by t, stretch/contract vector by a factor of t.

Ex: $3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$



* Reverse direction if t is negative *

Ex: $-\frac{1}{2} \cdot \begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$



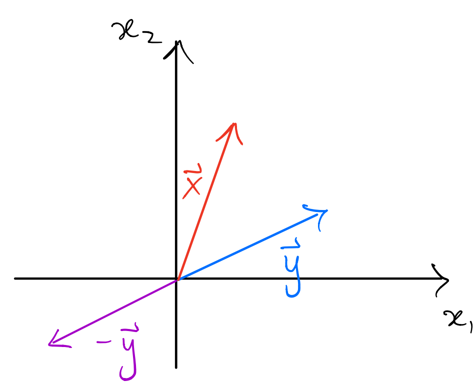
This also gives us a way to subtract!

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then

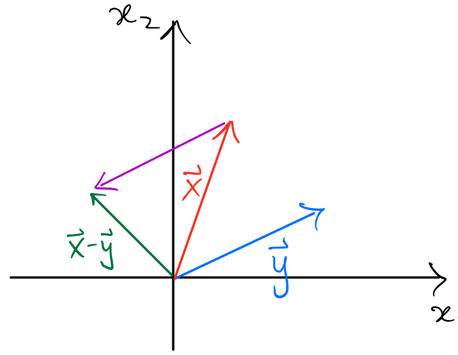
$$\boxed{\vec{x} - \vec{y} = \vec{x} + (-1)\vec{y} = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}}$$

Ex: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$

Graphically, reverse $\vec{y}$ . . .

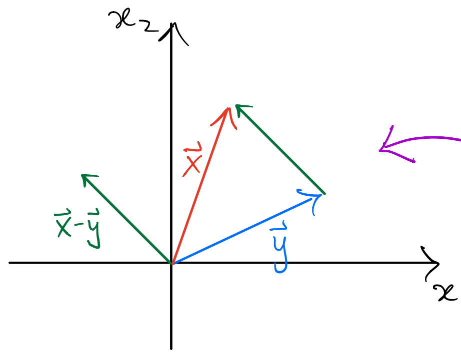


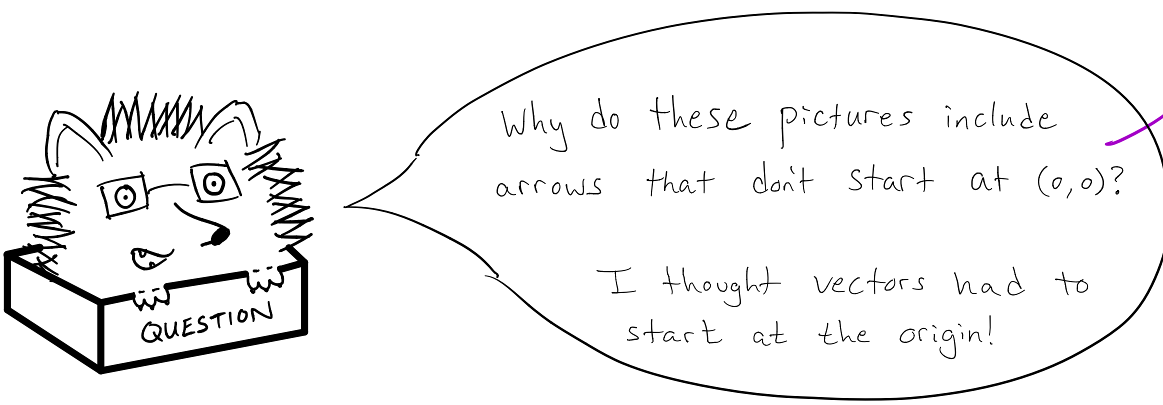
... and add to $\vec{x}$!



As we would expect,

$\vec{y} + (\vec{x}-\vec{y}) = \vec{x}$:

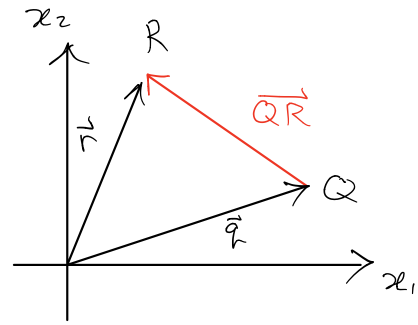




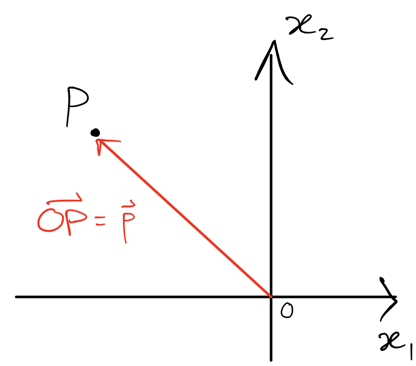
Vectors <u>do</u> start at the origin!

In many geometrical problems, however, it is useful to consider **directed line segments** that <u>don't</u> start at the origin.

Definition: The directed line segment from point Q to point R is denoted $\overrightarrow{QR}$.

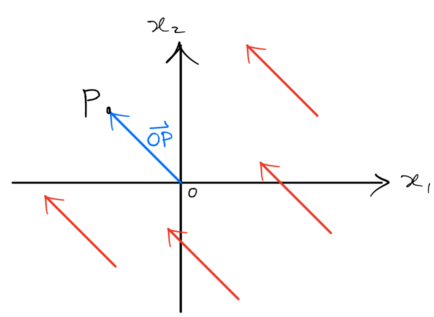


We will consider the directed line segment $\overrightarrow{OP}$ from the origin O to a point P to be the same as the vector $\vec{p}$.



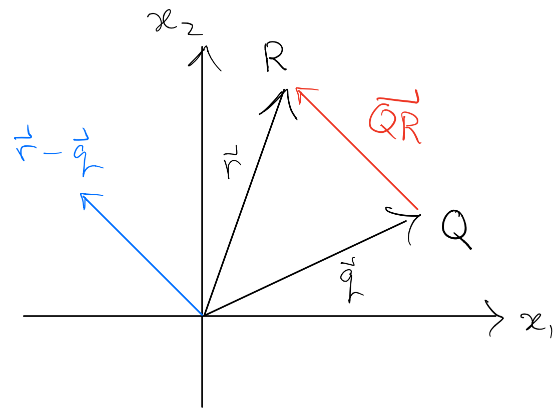
In many problems, we'll be interested in the length and direction of a directed line segment, but not its position.

So, we'll consider all of these directed line segments



to be the same as $\overrightarrow{OP} = \vec{p}$.

Thus, the line segment $\overrightarrow{QR}$ from $\vec{Q}$ to $\vec{R}$ is the same for us as $\vec{r}-\vec{q}$.



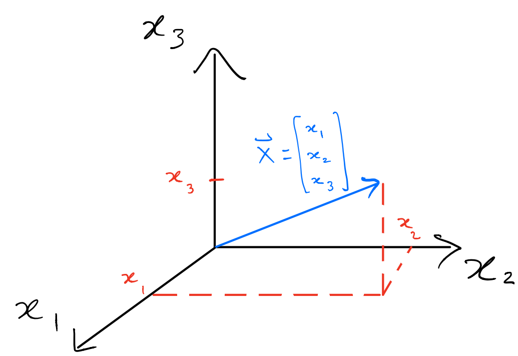
$$\boxed{\overrightarrow{QR} = \vec{r}-\vec{q}}$$

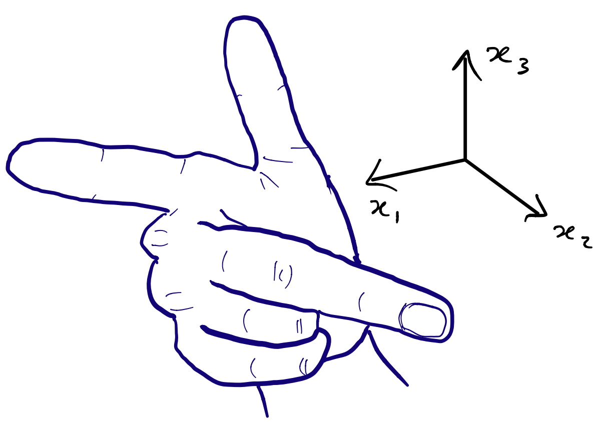
Vectors in $\mathbb{R}^n$

Everything above can also be done in higher dimensions:

$$\boxed{\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}}$$

We can visualize vectors in $\mathbb{R}^3$ just as we did in $\mathbb{R}^2$





Here, the three axes are oriented according to the right hand rule.

More generally, if n is any positive integer,

$$\boxed{\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}}$$

Definition: If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$

and $t \in \mathbb{R}$, then

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \qquad \text{(Addition)}$$

$$t\vec{x} = \begin{bmatrix} t x_1 \\ t x_2 \\ \vdots \\ t x_n \end{bmatrix} \qquad \text{(Scalar Multiplication)}$$

Ex: In $\mathbb{R}^4$, $2\begin{bmatrix} 3 \\ 0 \\ 1/2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \\ 8 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 7 \\ 6 \end{bmatrix}$

For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

(1) $\vec{x} + \vec{y} \in \mathbb{R}^n$ (closed under addition)
(2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (addition is commutative)
(3) $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ (addition is associative)
(4) There exists a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{z} + \vec{0} = \vec{z}$ for all $\vec{z} \in \mathbb{R}^n$ (zero vector)
(5) For each $\vec{x} \in \mathbb{R}^n$ there exists a vector $-\vec{x} \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverses)
(6) $t\vec{x} \in \mathbb{R}^n$ (closed under scalar multiplication)
(7) $s(t\vec{x}) = (st)\vec{x}$ (scalar multiplication is associative)
(8) $(s + t)\vec{x} = s\vec{x} + t\vec{x}$ (a distributive law)
(9) $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$ (another distributive law)
(10) $1\vec{x} = \vec{x}$ (scalar multiplicative identity)

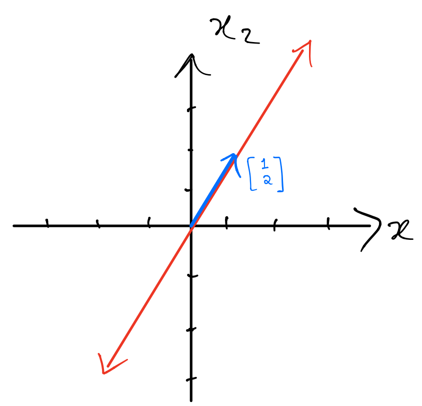
Properties of addition, scalar multiplication

Lines in $\mathbb{R}^n$

The set of vectors of the form

$$\boxed{\vec{x} = t\begin{bmatrix}1\\2\end{bmatrix}, \quad t \in \mathbb{R}}$$

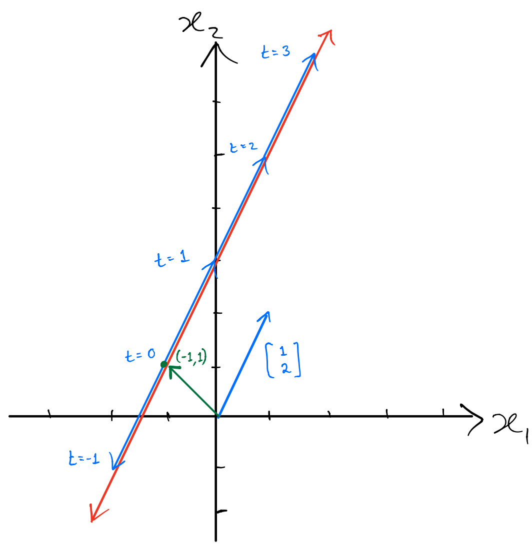
describes a line through the origin that moves in the same direction as $\begin{bmatrix}1\\2\end{bmatrix}$.



What about the set of vectors of the form

$$\boxed{\vec{x} = \begin{bmatrix}-1\\1\end{bmatrix} + t\begin{bmatrix}1\\2\end{bmatrix}, \quad t \in \mathbb{R}}\ ?$$

This new line passes through $(-1, 1)$, but still moves in the the same direction as $\begin{bmatrix}1\\2\end{bmatrix}$. (i.e, it is parallel to $\begin{bmatrix}1\\2\end{bmatrix}$.)



We could have also described each coordinate separately:

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}}$$

$$\Downarrow$$

$$\boxed{\begin{cases} x_1 = -1 + 1t \\ x_2 = 1 + 2t, \end{cases} \quad t \in \mathbb{R}}$$

Definition: Let $\vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$, $\vec{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$.

The vector equation of the line passing through P and in the direction of $\vec{d}$ is

$$\boxed{\vec{x} = \vec{p} + t\vec{d}, \quad t \in \mathbb{R}.}$$

The parametric equation of this line is

$$\boxed{\begin{cases} x_1 = p_1 + t d_1 \\ x_2 = p_2 + t d_2 \\ \vdots \qquad \vdots \\ x_n = p_n + t d_n \end{cases}, \quad t \in \mathbb{R}.}$$

· Two lines

$$\begin{cases} \vec{x} = \vec{p} + t\vec{d}, & t \in \mathbb{R} \\ \vec{y} = \vec{q} + t\vec{e}, & t \in \mathbb{R} \end{cases}$$

are parallel if $\vec{d}$ is a (non-zero) scalar multiple of $\vec{e}$.

Ex: Find the vector equation of the line

(a) through $P(1,0)$ and in the direction of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$;

(b) through $P(2,1)$ and parallel to the line $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -6 \\ 2 \end{bmatrix}$, $t \in \mathbb{R}$;

(c) through $P(0,3)$ and $Q(3,2)$.

Solution:

(a) $\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, so $\boxed{\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \ t \in \mathbb{R}.}$

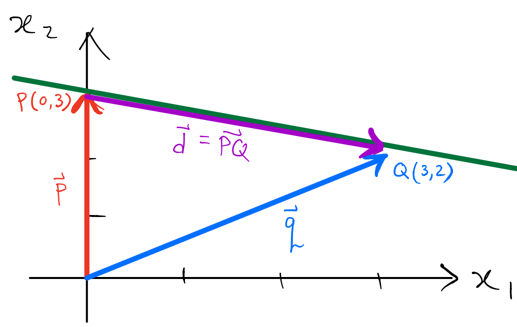
(b) Here, $\vec{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Since our line is parallel to

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -6 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}$$

$\vec{d}$ = any non-zero multiple of $\begin{bmatrix} -6 \\ 2 \end{bmatrix}$. Let's pick $\vec{d} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

Our line is $\boxed{\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -6 \\ 2 \end{bmatrix}, \ t \in \mathbb{R}.}$

(c) What's the direction vector? It's the line segment from $P(0,3)$ to $Q(3,2)$:



Sound familiar?

It's exactly

$$\vec{d} = \vec{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(note: $\vec{d} = \vec{QP}$ also works!)

We can use either P or Q as a point on the line.

One equation: $\boxed{\vec{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}.}$

Exercise: Are any of the lines from parts (a) - (c) above parallel to each other?

Explain.

Ex: Determine vector and parametric equations for the line through $P(4,0,1)$ and $Q(1,1,0)$ in $\mathbb{R}^3$.

Solution: Let's start with the vector equation:

Direction vector $\vec{d}$ moves from $P(4,0,1)$ to $Q(1,1,0)$, so

$$\vec{d} = \overrightarrow{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$$

We may use either P or Q as a point on the line.

One equation: $\boxed{\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}}$

Write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to get the parametric equation:

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$$

$$\implies \boxed{\begin{cases} x_1 = 4 - 3t \\ x_2 = t \\ x_3 = 1 - t \end{cases}, \quad t \in \mathbb{R}.}$$