Normal Limits of Nilpotents and Normal Operator Similarity Orbits in Purely Infinite C^*-Algebras

by

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Contents

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1 Introduction

An operator T in a C^*-algebra $\mathfrak{A}$ is called **nilpotent** if there exists a positive integer k such that $T^k = 0$. It has been of significant interest to mathematicians, especially over the past half century, to determine the norm closure of nilpotent operators in a C^*-algebra. The collective effort of several renowned mathematicians over multiples years led to a characterization of this closure in the setting of bounded linear operators acting on a complex, separable, infinite-dimensional Hilbert space in 1974 [1]. This characterization will be presented in Section 2, along with some of the fundamental results and techniques that were developed in the process.

In recent years, P. Skoufranis [22] demonstrated that similar necessary and sufficient conditions may be obtained to determine whether a normal operator belongs to the closure of nilpotents in a unital, simple, purely infinite C^*-algebra. These algebras are examined thoroughly in Section 3, and in particular, we shall see that the success observed by Skoufranis in this setting is due to the sheer abundance of projections that purely infinite C^*-algebras contain. A detailed analysis of Skoufranis' findings is presented in Section 4. One may hope that these results extend to general C^*-algebras, but as we demonstrate in Section 5, many obstructions arise when working in a C^*-algebra that possesses a faithful tracial state.

The latter half of this paper will showcase some of the recent results of P. Skoufranis [20] on unitary and similarity orbits of normal operators in unital, simple, purely infinite C^*-algebras. As is outlined in Section 6, Skoufranis constructed an operator theoretic proof of a result due to Dadarlat [5] which provides necessary conditions for two normal operators to be approximately unitarily equivalent. These results see application in Section 7, wherein we expose Skoufranis' approach to obtaining important bounds on the distance between unitary orbits of normal operators.

Finally, we shall bridge the gap between the approximation results of Section 4 and our discussion of normal operator unitary orbits in Sections 6 & 7. Since the normal limits of nilpotent operators in a unital, simple, purely infinite C^*-algebra are known, it becomes interesting to know what non-normal elements are also limits of nilpotents in these algebras. Of course, if an operator is a limit of nilpotents, then so too is any element within the closure of the similarity orbit of that operator, and hence it is tempting to ask which non-normal operators make up the closures of these orbits. Although the answer to this question is not currently known, using information on the distance between normal operator unitary orbits given in Section 7, Skoufranis ([20]) derived necessary and sufficient conditions for a normal operator to lie within the closed similarity orbit of a given normal element. A statement of this result is presented in Section 8, along with a detailed look at the approach taken by Skoufranis.

2 Approximation by Nilpotents in $\mathcal{B}(\mathcal{H})$

In 1970, P. R. Halmos published an influential article titled *Ten Problems in Hilbert Space* [10], which outlined some of the most important open problems facing operator theorists at that time. The seventh of these problems asked whether every quasinilpotent operator (that is, an operator whose spectrum is $\{0\}$) acting on a complex, separable, infinite-dimensional Hilbert space is a norm limit of nilpotent operators. Although not an answer to Halmos' problem, an example due to Kakutani showed that there exist non-quasinilpotent operators that are the norm limits of nilpotents. Thus, Halmos reposed his question in the following way: "What is the closure of nilpotent operators on a complex, separable, infinite-dimensional Hilbert space? In particular, does this closure contain all quasinilpotent operators?"

Many esteemed mathematicians approached this problem over the next several years, but it was in 1973 that Herrero [11] made a fundamental advancement. Namely, he proved that a normal operator is a norm limit of nilpotent operators if and only if its spectrum is connected and contains 0. Herrero's contribution was a major step in the characterization of the closure of nilpotents by Apostol, Foiaş, and Voiculescu [1] in 1974, which provided a solution to Halmos' revised question.

It is the goal of this section to highlight some of the key ingredients used in obtaining the solution to Halmos' seventh problem. An investigation of Berg's technique (a method for intertwining two forward weighted shifts) and its consequences in Section 2.1 will allow us to outline a short proof Herrero's theorem in Section 2.2. Following this exposition, Section 2.3 will examine some characterization theorems for the closure of nilpotent operators in certain C^*-algebras (including the previously mentioned result of Apostol, Foiaş, and Voiculescu), and provide motivation for new directions in which we may extend these results.

2.1 Berg's Technique

Berg's technique is a method for intertwining two forward weighted shifts without greatly disrupting the norm. The idea is to introduce a small twist that executes this intertwining over a certain number of steps. This twist works by applying subtle rotations on orthogonally acting components of each shift. Since these components are orthogonal to one another, the effects of the perturbations are not compounded, but rather the overall difference in norm is given by the maximum of the individual changes. Figure 1. depicts this situation, where the diagonal lines represent the effects of the twist operator, and the vertices represent the bases $\{e_j\}$ and $\{f_j\}$ described below.

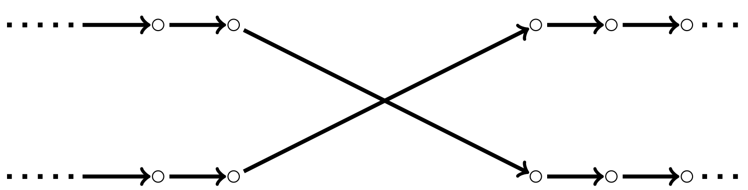

Figure 1: Berg's Technique

As we shall see, the total change in norm decreases as the interval over which the twist occurs increases. A formal statement of this result is presented in the following theorem.

Theorem 2.1.1 (Berg's Technique [3])**.** *Let $\{e_j, f_j : j = 0, \ldots, m\}$ be an orthonormal family in a Hilbert space $\mathcal{H}$. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear map satisfying*

$$Te_j = e_{j+1}\,, \ \text{and} \ \ Tf_j = f_{j+1}$$

for every $j \in \{0, \ldots, m-1\}$*. Then there exists* $S \in \mathcal{B}(\mathcal{H})$ *such that*

(1) $Sx = Tx$ *for all* $x \in \{e_i, f_j : j = 0, \ldots, m-1\}^\perp$,

(2) $S(\text{span}\{e_j, f_j\}) = \text{span}\{e_{j+1}, f_{j+1}\}$ *for all* $j \in \{0, \ldots, m-1\}$,

(3) S *is an isometry on* $\text{span}\{e_j, f_j\}$ *for all* $j \in \{0, \ldots, m-1\}$,

(4) $S^m e_0 = f_m$, $S^m f_0 = e_m$, *and*

(5) $\|S - T\| < \pi/m$.

Proof. For each $j \in \{0, \ldots, m\}$, let $\mathscr{M}_j := \text{span}\{e_j, f_j\}$. Note that for each $j \in \{0, \ldots, m-1\}$, the operator $T \restriction_{\mathscr{M}_j}$ is a unitary U_j taking $\mathscr{M}_j$ onto $\mathscr{M}_{j+1}$, and in the given bases,

$$U_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define the operator

$$\theta := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/m} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

and note that

- θ is unitary,
- $\|\theta - I\| = |e^{i\pi/m} - 1| < \pi/m$, and
- $\theta^m = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We are now prepared to define the operator S. If $x \in \{e_j, f_j : 0 \leq j \leq m-1\}^\perp$, then define $Sx := Tx$ so that condition (1) is satisfied. For each $j \in \{0, \ldots, m-1\}$, define S to be the unitary θU_j from $\mathscr{M}_j$ onto $\mathscr{M}_{j+1}$ and extend S linearly to all of $\mathcal{H}$. In this way, it is immediate that conditions (2) and (3) are satisfied by S. Notice also that S^m takes $\mathscr{M}_0$ onto $\mathscr{M}_m$ via the matrix θ^m. From the description of θ^m given above, it is evident that $S^m e_0 = f_m$ and $S^m f_0 = e_m$, and thus S satisfies condition (4). Finally, observe that the restriction of $S - T$ to $\mathscr{M}_0 \oplus \cdots \oplus \mathscr{M}_m$ is represented by the matrix

$$\begin{bmatrix} 0 & & & \\ (\theta - I) & 0 & & \\ & \ddots & \ddots & \\ & & (\theta - I) & 0 \end{bmatrix}$$

in the given bases. Since S and T agree on $\{e_j, f_j : 0 \leq j \leq m-1\}^\perp$, it follows that

$$\|S - T\| = \|\theta - I\| < \pi/m,$$

and condition (5) is satisfied. ∎

2.2 Herrero's Theorem

A fundamental breakthrough in solving Halmos' seventh problem was seen in the following result due to Herrero, which characterizes which normal operators in $\mathcal{B}(\mathcal{H})$ are norm limits of nilpotent operators. We will use the notation $\mathrm{Nil}(\mathfrak{A})$ and $\mathrm{QNil}(\mathfrak{A})$ to describe the nilpotent elements and quasinilpotent elements in a C^*-algebra $\mathfrak{A}$, respectively.

Theorem 2.2.1 (Herrero [11])**.** *Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space, and let N be a normal element in $\mathcal{B}(\mathcal{H})$. Then $N \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ if and only if $\sigma(N)$ is connected and contains 0.*

Note that the necessity of these conditions can be seen as a quick consequence of the upper semicontinuity of the spectrum. Although the original proof that demonstrates the sufficiency of such conditions is somewhat technical, a more recent argument (also due to Herrero) has been constructed as an application of Berg's technique. In particular, Berg's technique can be used to demonstrate the existence of a sequence of normal operators in $\mathcal{B}(\mathcal{H})$ whose distance to the set of nilpotents tends to 0, and whose spectra are asymptotically dense in the closed unit disk, $\overline{\mathbb{D}}$. This fact is reproduced below and is based on the proof presented in [21].

Lemma 2.2.2. *Let $m, n \in \mathbb{N}$ with $m \geq 2$. Then there exists a nilpotent matrix $M \in \mathbb{M}_{2(m+1)n+1}(\mathbb{C})$ and and normal matrix $N \in \mathbb{M}_{2(m+1)n+1}(\mathbb{C})$ such that*

(1) $\|M - N\| \leq \frac{\pi}{n} + \frac{1}{m}$, *and*

(2) $\sigma(N) = \left\{ \frac{k}{m} e^{\frac{i\pi}{n} j} : j = 1, \ldots, 2n\,;\, k = 0, \ldots, m \right\}$, *where the multiplicity of the zero eigenvalue is $n+1$, and the multiplicity of every other eigenvalue is 1.*

Proof. We begin by defining the nilpotent matrix $M \in \mathbb{M}_{2(m+1)n+1}$ to be a particular forward weighted shift. For each $k \in \{0, \ldots, m\}$, define $a_k := k/m$, and let

$$\{e_k : k = 0, \ldots, (2m+1)n\}$$

denote the standard orthonormal basis for $\mathbb{C}^{(2m+1)n+1}$. We then define M on $\mathbb{C}^{(2m+1)n+1}$ by setting

- $M(e_{kn+j}) = a_{k+1} e_{kn+j+1}$ for all $k \in \{0, 1, \ldots, m-1\}$ and $j \in \{0, \ldots, n-1\}$,
- $M(e_{mn+j}) = a_m e_{mn+j+1}$ for all $j \in \{0, \ldots, n-1\}$,
- $M(e_{kn+j}) = a_{2m+1-k} e_{kn+j+1}$ for all $k \in \{m+1, \ldots, 2m\}$ and $j \in \{0, \ldots, n-1\}$,
- $M e_{(2m+1)n} = 0$,

and extending by linearity. If, for each $k \in \{0, \ldots, 2m-1\}$, we define

$$\mathcal{B}_k := \{e_{nk+j} : j = 0, \ldots, n-1\},$$

and $\mathcal{H}_k := \mathrm{span}(\mathcal{B}_k)$, then the weight sequence of M can be represented by the diagram in Figure 2. The basis vectors are represented along the x-axis, and their corresponding weights on the y-axis.

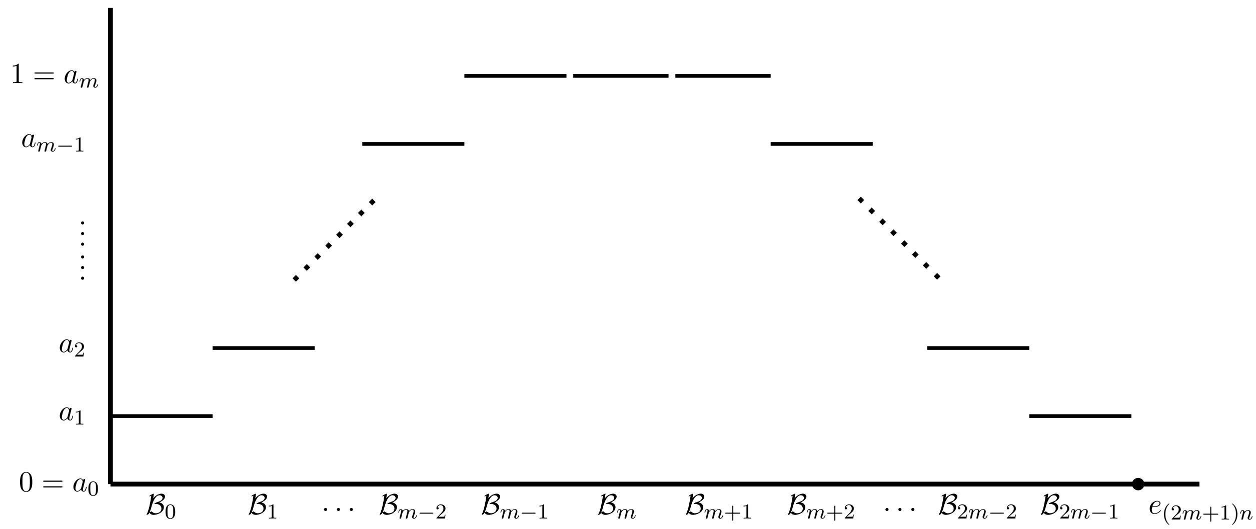

Figure 2: Nilpotent Forward Weighted Shift - M

Let $\mathcal{H}$ be any Hilbert space with orthonormal basis given by $\{f_1, \ldots, f_{2n}\}$. Define the operator U_{2n} on $\mathcal{H}$ to be the cycle with weights 1, so

$$U_{2n} = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{2n \times 2n}.$$

in the given basis. We note at this point that U_{2n} is a unitary operator with spectrum given by the $(2n)^{th}$-roots of unity, each with multiplicity 1. As we will see, we can approximate M with a certain direct sum of multiples of U_{2n} via Berg's technique.

Let $\mathcal{K}_{m-1,m+1} := \mathcal{H}_{m-1} \oplus \mathcal{H}_m \oplus \mathcal{H}_{m+1}$. By applying Berg's technique on $\{e_{nm-n}, \ldots, e_{nm}\}$ and $\{e_{nm+n}, \ldots, e_{nm+2n}\}$, we obtain a matrix $S_1 \in \mathbb{M}_{(2m+1)n+1}(\mathbb{C})$ such that

- $S_1 x = Mx$ for all $x \in (\mathcal{H}_{m-1} \oplus \mathcal{H}_{m+1})^\perp$,
- $S_1(\text{span}\{e_{nm-n+j}, e_{nm+n+j}\}) = \text{span}\{e_{nm-n+j+1}, e_{nm+n+j+1}\}$ for all $j \in \{0, \ldots, n-1\}$,
- S_1 is an isometry on $\text{span}\{e_{nm-n}, \ldots, e_{nm-1}, e_{nm+n}, \ldots, e_{nm+2n-1}\}$,
- $S_1^n(e_{nm-n}) = e_{nm+2n}$ and $S_1^n(e_{nm+n}) = e_{nm}$, and
- $\|S_1 - M\| < \frac{\pi}{n}$.

It is then clear that $\mathcal{K}'_{m-1,m+1} := \text{span}\{e_{nm+n}, S_1 e_{nm+n}, \ldots, S_1^{n-1} e_{nm+n}\} \oplus \mathcal{H}_m$ is a reducing subspace for S_1, and the restriction of S_1 to $\mathcal{K}'_{m-1,m+1}$ is unitarily equivalent to U_{2n}. By letting

$$\mathcal{K}''_{m-1,m+1} := \mathcal{K}_{m-1,m+1} \ominus \mathcal{K}'_{m-1,m+1} = \text{span}\{e_{nm-n}, S_1 e_{nm-n}, \ldots, S_1^{n-1} e_{nm-n}\},$$

we see that the restriction of S_1 to $\mathcal{K}''_{m-1,m+1}$ is a forward shift with weights 1, and

$$S_1(e_{nm+2n}) = M(e_{nm+2n}) = a_{m-1} e_{nm+2n+1}.$$

Further, if we let M_1 denote the operator S_1 with the weights on $\mathcal{K}''_{m-1,m+1}$ reduced from 1 to a_{m-1}, then

$$\|M - M_1\| \leq \|M - S_1\| + \|S_1 - M_1\| \leq \frac{\pi}{n} + \frac{1}{m},$$

and with respect to the decomposition $\mathcal{K}'_{m-1,m+1} \oplus (\mathcal{K}'_{m-1,m+1})^{\perp}$, we may write $M_1 = U_{2n} \oplus M'_1$, where $M' \in \mathbb{M}_{(2m-1)n+1}$ is a nilpotent forward weighted shift with weights

$$a_1, a_2, \ldots, a_{m-2}, a_{m-1}, a_{m-1}, a_{m-1}, a_{m-2}, \ldots, a_2, a_1$$

corresponding to segments of length n. That is, M' decomposes as a $2n \times 2n$ loop with weights 1, and a nilpotent forward weighted shift that is of a form similar to that of the operator in Figure 2, but with the weights reduced by a factor of $1/m$ on one of the n-blocks.

For our next step, we define $\mathcal{K}_{m-2,m+2} := \mathcal{H}_{m-2} \oplus \mathcal{K}''_{m-1,m+1} \oplus \mathcal{H}_{m+2}$, and apply Berg's technique on $\{e_{nm-2n}, \ldots, e_{nm-n}\}$ and $\{e_{nm+2n}, \ldots, e_{nm+3n}\}$. As in in the previous case, we obtain a matrix $S_2 \in \mathbb{M}_{(2m+1)n+1}(\mathbb{C})$ such that

- $S_2 x = M_1 x$ for all $x \in (\mathcal{H}_{m-2} \oplus \mathcal{H}_{m+2})^{\perp}$,
- $S_2(\mathrm{span}\{e_{nm-2n+j}, e_{nm+2n+j}\}) = \mathrm{span}\{e_{nm-2n+j+1}, e_{nm+2n+j+1}\}$ for all $j \in \{0, \ldots, n-1\}$,
- S_2 is a_{m-1} times an isometry on $\mathrm{span}\{e_{nm-2n}, \ldots, e_{nm-n-1}, e_{nm+2n}, \ldots, e_{nm+3n-1}\}$,
- $S_2^n(e_{nm-2n}) = a_{m-1}^n e_{nm+3n}$ and $S_2^n(e_{nm+2n}) = a_{m-1}^n e_{nm-n}$, and
- $\|S_2 - M_1\| < \frac{\pi}{n}$.

It is then clear $\mathcal{K}'_{m-2,m+2} := \mathrm{span}\{e_{nm+2n}, S_2 e_{nm+2n}, \ldots, S_2^{n-1} e_{nm+2n}\} \oplus \mathcal{K}''_{m-1,m+1}$ is a reducing subspace for S_2, and the restriction of S_2 to $\mathcal{K}'_{m-2,m+2}$ is unitarily equivalent to $a_{m-1}U_{2n}$. Moreover, S_2 is a forward shift with with weights a_{m-1} when restricted to $\mathcal{K}''_{m-2,m+2} := \mathcal{K}_{m-2,m+2} \ominus \mathcal{K}'_{m-2,m+2}$. Again, we may reduce the weights on $\mathcal{K}''$ by a factor of $1/m$ and arrive at a matrix M_2 such that with respect to the decomposition

$$\mathcal{K}'_{m-1,m+1} \oplus \mathcal{K}'_{m-2,m+2} \oplus (\mathcal{K}'_{m-1,m+1} \oplus \mathcal{K}'_{m-2,m+2})^{\perp},$$

M_2 has the form $U_{2n} \oplus a_{m-1}U_{2n} \oplus M'_2$, where M'_2 is the $(n+1) \times (n+1)$ zero matrix if $m = 2$, and $M'_2 \in \mathbb{M}_{(2m-3)n+1}(\mathbb{C})$ is a nilpotent forward weighted shift with weights

$$a_1, a_2, \ldots, a_{m-3}, a_{m-2}, a_{m-2}, a_{m-2}, a_{m-3}, \ldots, a_2, a_1$$

corresponding to segments of length n, otherwise. At this point we note that M_1 was obtained from M by only its values of $\mathcal{K}''_{m-1,m+1}$, and M_2 was obtained from M_1 by perturbing only its values on $\mathcal{K}''_{m-2,m+2}$. Since these two spaces are orthogonal, and $\|M_1 - M_2\| \leq \frac{\pi}{n} + \frac{1}{m}$, it follows that

$$\|M - M_2\| \leq \max\{\|M - M_1\|, \|M_1 - M_2\|\} \leq \frac{\pi}{n} + \frac{1}{m}.$$

This process may now be repeated a finite number of times to arrive at an operator N that is unitarily equivalent to

$$U_{2n} \oplus a_{m-1}U_{2n} \oplus \cdots \oplus a_1 U_{2n} \oplus 0_{(n+1)\times(n+1)}.$$

Hence, N is a normal operator and our knowledge of the spectrum of U_{2n} implies that $\sigma(N)$ is as advertised. Finally, the above construction implies that $\|N - M\| \leq \frac{\pi}{n} + \frac{1}{m}$, and the proof is complete.

■

Figure 3. provides a visual representation of the spectrum of the normal operator from Lemma 2.2.2 for different values of m and n. As claimed, we see that the spectrum of this operator "fills out" $\overline{\mathbb{D}}$ as m and n become large.

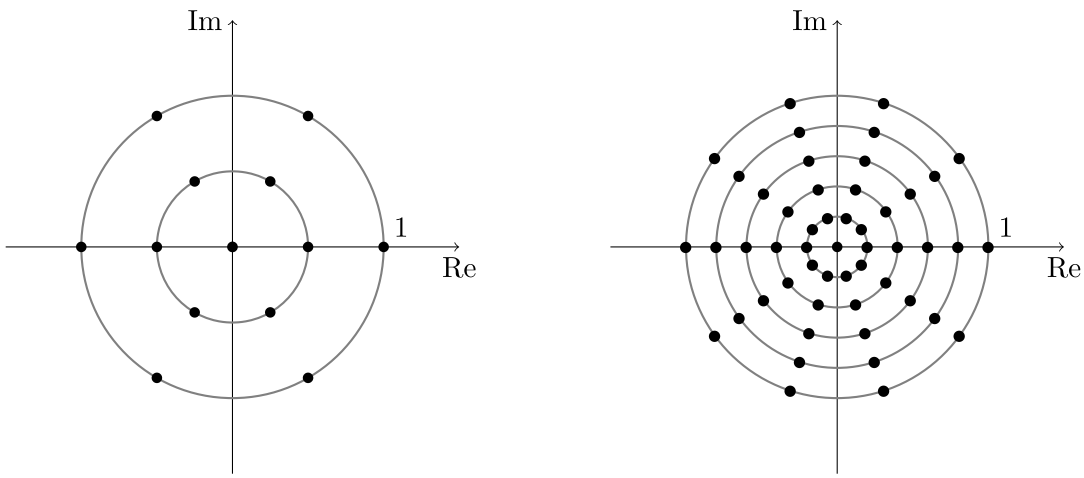

Figure 3: $\sigma(N)$ where $m = 2, n = 3$ (left) and $m = 5, n = 5$ (right)

Lemma 2.2.2 now allows one to prove the following special case of Theorem 2.2.1:

Theorem 2.2.3. *Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space. If D is a normal operator in $\mathcal{B}(\mathcal{H})$ and $\sigma(D) = \overline{\mathbb{D}}$, then $D \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$.*

Proof. Let $\varepsilon > 0$ and let $\mathcal{D} = \{d_i\}_{i=1}^{\infty}$ be a countable dense subset of $\overline{\mathbb{D}}$. For each $k \in \mathbb{N}$, let N_k and M_k denote the normal operator and nilpotent operator from Lemma 2.2.2, respectively, with $m = n = k$. Choose n large enough so that $\mathcal{U} := \{B_\varepsilon(\lambda) : \lambda \in \sigma(N_n)\}$ forms an open cover of $\overline{\mathbb{D}}$, and such that $5/n < \varepsilon$. Let $N_n^{(\infty)}$ and $M_n^{(\infty)}$ denote the direct sums of infinitely many copies of N_n and M_n, respectively, so that

- $\sigma\left(N_n^{(\infty)}\right) = \sigma(N_n)$, now with every element possessing infinite multiplicity,
- $N_n^{(\infty)}$ is normal,
- $M_n^{(\infty)}$ is nilpotent, and
- $\|N_n^{(\infty)} - M_n^{(\infty)}\| = \|N_n - M_n\| < 5/n < \varepsilon$.

As a normal matrix of finite dimension, each N_n is diagonalizable. It follows that $N_n^{(\infty)}$ is also diagonalizable, and hence we may choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{H}$ so that

$$N_n^{(\infty)} = \bigoplus_{i=1}^{\infty} \mathrm{diag}(\lambda_1, \ldots, \lambda_k),$$

where $\lambda_1, \ldots, \lambda_k$ are the distinct elements from $\sigma(N_n)$. Since $\mathcal{U}$ is an open cover for $\overline{\mathbb{D}}$, we may associate to each element $d_i \in \mathcal{D}$ a member $\lambda(i) \in \sigma(N_n)$ so that $d_i \in B_\varepsilon(\lambda(i))$. Further, we may assume that each $\lambda \in \sigma(N_n)$ occurs as a $\lambda(i)$ for infinitely many values of i, as each $B_\varepsilon(\lambda)$ contains infinitely many elements of $\mathcal{D}$.

Define the operator $D' : \mathcal{H} \to \mathcal{H}$ by

$$D'e_i = d_i e_i,$$

and note that, by applying a basis permutation if necessary, we may assume that

$$N_n^{(\infty)} = \begin{bmatrix} \lambda(1) & & & \\ & \lambda(2) & & \\ & & \lambda(3) & \\ & & & \ddots \end{bmatrix},$$

and hence $\|N_n^{(\infty)} - D'\| < \varepsilon$. Moreover, D' is diagonal and hence normal, and $\sigma(D') = \overline{\mathbb{D}} = \sigma(D)$, by construction. That being said, the Weyl–von Neumann–Berg Theorem (see [9, Corollary II.4.2]) implies that D' is approximately unitarily equivalent to D. That is, there is a sequence $\{U_m\}_{m=1}^{\infty}$ of unitaries such that

$$\lim_{m\to\infty} \|U_m D' U_m^* - D\| = 0.$$

Choosing $m' \in \mathbb{N}$ large enough so that $\|U_m D' U_m^* - D\| < \varepsilon$ whenever $m \geq m'$, it quickly follows that

$$\|D - U_m M_n^{(\infty)} U_m^*\| = \|D - U_m D' U_m^*\| + \|D' - N_n^{(\infty)}\| + \|N_n^{(\infty)} - M_n^{(\infty)}\| < 3\varepsilon$$

for all $m \geq m'$. Since $U_m M_n^{(\infty)} U_m^*$ is nilpotent for every $m \in \mathbb{N}$, we conclude that $D \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$, and the proof is complete.

■

We now examine a simple but important fact that will be useful in proving Herrero's theorem. This result will also be applied heavily throughout Sections 4, 6, and 7. Here we shall introduce the notation $\mathrm{Nor}(\mathfrak{A})$ to denote the set of normal elements in a C^*-algebra $\mathfrak{A}$.

Lemma 2.2.4. *Let $\mathfrak{A}$ be a C^*-algebra, $N \in \mathrm{Nor}(\mathfrak{A})$, and $(N_n)_{n\geq 1}$ be a sequence of normal operators in $\mathfrak{A}$ whose limit is N. If $U \subseteq \mathbb{C}$ is open and $\sigma(N) \cap U \neq \emptyset$, then there exists $k \in \mathbb{N}$ such that $\sigma(N_n) \cap U \neq \emptyset$ whenever $n \geq k$.*

Proof. Suppose that $\sigma(N_n) \cap U = \emptyset$ for infinitely many values of n, and let $\lambda \in \sigma(N) \cap U$. By Urysohn's lemma, there exists a continuous function f on $\mathbb{C}$ such that $f(\lambda) = 1$ and f is identically 0 when restricted to $\mathbb{C} \setminus U$. Further, for all $n \in \mathbb{N}$ with $\sigma(N_n) \cap U = \emptyset$, we have that $f(N_n) = 0$ by choice of f. Since this occurs for infinitely many values of n and

$$f(N_n) \to f(N),$$

it follows that $f(N) = 0$. This cannot be the case, however, as $\lambda \in \sigma(N)$ and $f(\lambda) = 1$.

■

With these results in hand we are but a stone's throw from Herrero's theorem. Note that if p is a polynomial that vanishes at 0, then $p(N)$ is a norm limit of nilpotents operators in $\mathcal{B}(\mathcal{H})$ whenever N is. It is then straightforward to verify that for such N, $f(N) \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ whenever f is a function that is analytic on $\sigma(N)$, and $f(0) = 0$. The remainder of the proof will proceed as follows.

Proof of Theorem 2.2.1. Let $N \in \mathcal{B}(\mathcal{H})$ be such that $\sigma(N)$ is connected and contains 0, and let $\varepsilon > 0$. One may find a simply connected open set U that contains $\sigma(N)$ and satisfies

$$\operatorname{dist}(z, \sigma(N)) \leq \varepsilon$$

for all $z \in \overline{U}$. If we let $\mathbb{D}_{1+\varepsilon}$ denote the open disk in $\mathbb{C}$ with center 0 and radius $1+\varepsilon$, then by the Riemann mapping theorem we obtain an analytic bijection

$$f : \mathbb{D}_{1+\varepsilon} \to U.$$

It is evident that we may compose f with a Möbius transformation if necessary and assume that $f(0) = 0$.

Note that since f is continuous and $\sigma(N)$ is compact, $f^{-1}(N)$ defines a compact subset of $\mathbb{D}_{1+\varepsilon}$. Hence, there exists $r < 1 + \varepsilon$ such that $f^{-1}(U) \subseteq \overline{\mathbb{D}_r}$. Let N' be any normal operator in $\mathcal{B}(\mathcal{H})$ with spectrum equal to $\overline{\mathbb{D}_r}$, and note that $N' \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$. To see this, simply observe that $r^{-1}N' \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ by Theorem 2.2.3, and hence $N' \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ as well. It follows from the remarks preceding this proof that $f(N')$ is a normal element of $\mathcal{B}(\mathcal{H})$ contained in $\overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$, and

$$\sigma(f(N')) = f(\sigma(N')) = f(\overline{\mathbb{D}_r}) \supseteq f\left(f^{-1}(\sigma(N))\right) = \sigma(N).$$

Moreover, it is clear that $f(\sigma(N')) \subseteq U$, and hence $\sigma(N) \subseteq \sigma(f(N')) \subseteq U$. Let $\chi_{\sigma(N)}$ denote the characteristic function on $\sigma(N)$, and define

$$N'' := \chi_{\sigma(N)}(f(N'))f(N').$$

We notice that N'' is a normal operator in $\mathcal{B}(\mathcal{H})$ with $\sigma(N'') = \sigma(N)$, and $\|N'' - f(N')\| \leq \varepsilon$. Further, since $f(N')$ is a norm limit of nilpotent operators, it is clear that

$$\operatorname{dist}(N'', \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}) \leq \|N'' - f(N')\| + \operatorname{dist}(f(N'), \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}) \leq \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we may construct a sequence of normal operators $(M_n)_{n\geq 1}$ in $\mathcal{B}(\mathcal{H})$ with $\sigma(M_n) = \sigma(N)$ for every $n \in \mathbb{N}$, and such that $\lim_{n\to\infty} \operatorname{dist}(M_n, \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}) = 0$. If we let M denote the limit of this sequence, then M is a normal element of $\mathcal{B}(\mathcal{H})$ that is a norm limit of nilpotent operators, and $\sigma(M) = \sigma(N)$ by the upper semicontinuity of the spectrum and Lemma 2.2.4. The Weyl–von Neumann–Berg Theorem now implies that M and N are approximately unitarily equivalent, and hence $N \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$, as desired.

■

2.3 Characterization of $\overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ and $\overline{\mathrm{Nil}(\mathcal{Q}(\mathcal{H}))}$

Recall that for a Hilbert space $\mathcal{H}$, the Calkin algebra is defined as $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ is the closed ideal of $\mathcal{B}(\mathcal{H})$ consisting of all compact operators. Further, if $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ is the canonical quotient map, then the **essential spectrum** of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined to be $\sigma(\pi(T))$ and is denoted by $\sigma_e(T)$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called **semi-Fredholm** if it has closed range and at least one of $\ker T$ or $\ker T^*$ is finite-dimensional. We let

$$\rho_{sF}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is semi-Fredholm}\},$$

and if T is semi-Fredholm, we define the **semi-Fredholm index** of T by

$$\operatorname{ind}(T) := \dim\ker T - \dim\ker T^* \in \mathbb{Z} \cup \{\pm\infty\}.$$

The following characterization of $\overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ for a complex, separable, infinite-dimensional Hilbert space $\mathcal{H}$ was proven by Apostol, Foiaş, and Voiculescu in 1974. We shall simply state this result, but direct the reader to either [1] or [12] for a proof.

Theorem 2.3.1 (Apostol, Foiaş, Voiculescu [1]). *Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ belongs to $\overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ if and only if*

(1) *$\sigma(T)$ is connected and contains 0,*

(2) *$\sigma_e(T)$ is connected and contains 0, and*

(3) *$\mathrm{ind}(T - \lambda I) = 0$ for all $\lambda \in \rho_{sF}(T)$.*

As a consequence, the following corollary (the proof of which is trivial by Theorem 2.3.1) provides an affirmative answer to Halmos' original question:

Corollary 2.3.2. *If $\mathcal{H}$ is a complex, separable, infinite-dimensional Hilbert space, then*

$$\mathrm{QNil}(\mathcal{B}(\mathcal{H})) \subseteq \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}.$$

One may now ask whether these characterizations extend to other types of C^*-algebras. While no characterization of the closure of the nilpotents exists for a general C^*-algebra, some advancements have been made in special cases. We begin with a simple observation regarding abelian C^*-algebras.

Proposition 2.3.3. *If $\mathfrak{A}$ is a commutative C^*-algebra, then $\mathrm{QNil}(\mathfrak{A}) = \{0\}$.*

Proof. This result is trivial by the Gelfand–Naimark theorem, which states that if $A \in \mathfrak{A}$, then $\sigma(A)$ is simply the image of $\widehat{A}$ in the unital case, and the image of $\widehat{A}$ together with $\{0\}$ in the non-unital case. Since the Gelfand transform is injective, we have that $A = 0$ whenever $\sigma(A) = \{0\}$. ∎

More interesting is the case of the Calkin algebra. This result is summarized below, and a proof may be found in [12, Theorem 5.34].

Theorem 2.3.4. *Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space, and let T be an operator in $\mathcal{B}(\mathcal{H})$. Then $\pi(T) \in \overline{\mathrm{Nil}(\mathcal{Q}(\mathcal{H}))}$ is and only if*

(1) *$\sigma_e(T)$ is connected and contains 0, and*

(2) *$\mathrm{ind}(T - \lambda I) = 0$ for all $\lambda \in \rho_{sF}(T)$.*

As we will see in the next section, the Calkin algebra is an example of a more general class of C^*-algebras, known as *purely infinite* C^*-algebras. It is then natural to ask whether the results of this section extend to other C^*-algebras with this property.

3 Purely Infinite C^*-Algebras

Having obtained a characterization of $\overline{\text{Nil}(\mathcal{B}(\mathcal{H}))}$ from Section 2, one may ask if analogous results exist for other classes of C^*-algebras. Indeed, as we shall see in the coming sections, simple C^*-algebras that are so-called *purely infinite* enjoy many similar characterization results. We will observe that such characterizations can be achieved because of the abundance of projections these algebras contain.

It is necessary to obtain a firm understanding of purely infinite C^*-algebras before moving to approximation theory, and hence the goal of this section is to build the foundations required to extend the results of Section 2. As these algebras are defined by their hereditary C^*-subalgebras, Section 3.1 outlines a thorough examination of such objects based largely on the exposition in [14]. This leads us to Section 3.2, wherein we explore the structure of purely infinite C^*-algebras via an analysis of the projections within them. In particular, a useful algebraic characterization of such algebras is presented in Theorem 3.2.11, which can be used to show that the Calkin algebra is purely infinite. Finally, Section 3.3 demonstrates that simple, purely infinite C^*-algebras have *real rank zero*. The consequences of this fact play a key role in the sections to come.

3.1 Hereditary C^*-Subalgebras

Definition 3.1.1. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. We say $\mathfrak{B}$ is* ***hereditary*** *if whenever $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ with $0 \leq A \leq B$, we have $A \in \mathfrak{B}$.*

The purpose of this section is to obtain an understanding of the structure of hereditary C^*-subalgebras, and explore some important examples. As we shall see, such objects inherit many enjoyable properties from the parent C^*-algebra (see Corollary 3.1.8 for one such result.) First, we show that the following important classes of C^*-subalgebras are hereditary.

Proposition 3.1.2. *Let $\mathfrak{A}$ be a C^*-algebra and P be a projection in $\mathfrak{A}$. Then the C^*-subalgebra $P\mathfrak{A}P$ is hereditary. Such a C^*-subalgebra is called a* ***corner*** *of $\mathfrak{A}$.*

Proof. Let $A \in \mathfrak{A}$ and $B \in \mathfrak{A}$ be such that $0 \leq A \leq PBP$. By passing to the unitization of $\mathfrak{A}$, we see that

$$0 \leq (I-P)A(I-P) \leq (I-P)PBP(I-P) = 0,$$

and so $(I-P)A(I-P) = 0$. The C^*-identity now implies that

$$\|A^{1/2}(I-P)\|^2 = \|(I-P)A(I-P)\| = 0,$$

and hence $A^{1/2}(I-P) = 0$. Thus, $A(I-P) = 0 = (I-P)A$, and we may deduce that

$$A = PAP \in P\mathfrak{A}P.$$

This proves that $P\mathfrak{A}P$ is hereditary, as claimed. ■

Proposition 3.1.3. *Let $\mathfrak{A}$ be a C^*-algebra and $A \in \mathfrak{A}_{sa}$. Then $\mathfrak{B} := \overline{A\mathfrak{A}A}$ is a hereditary C^*-subalgebra of $\mathfrak{A}$ containing A.*

Proof. It is clear that $\mathfrak{B}$ is a C^*-subalgebra of $\mathfrak{A}$. Before proving that $\mathfrak{B}$ is hereditary, we first claim that if $X \in \mathfrak{B}_+ := \{B \in \mathfrak{B} : B \geq 0\}$, then $\overline{X\mathfrak{A}X} \subseteq \mathfrak{B}$. Indeed, if $X = \lim_n AX_nA \in \mathfrak{B}$ and $Y \in \mathfrak{A}$, then

$$XYX = \lim_{n\to\infty}(AX_nA)Y(AX_nA) \in \mathfrak{B}.$$

This proves that $X\mathfrak{A}X \subseteq \mathfrak{B}$, and since $\mathfrak{B}$ is closed, the proof of the claim is complete. Now to see heredity, suppose that $C \in \mathfrak{A}$ and $B \in \mathfrak{B}$ are such that $0 \leq C \leq B$. If $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $\mathfrak{B}$, then an application of the C^*-identity demonstrates that for each λ,

$$\begin{aligned}\|C^{1/2} - C^{1/2}e_\lambda\|^2 &= \|(I-e_\lambda)C(I-e_\lambda)\| \\ &\leq \|(I-e_\lambda)B(I-e_\lambda)\| \\ &= \|B^{1/2} - B^{1/2}e_\lambda\|^2.\end{aligned}$$

Note that $(I-e_\lambda)C(I-e_\lambda)$ is simply a shorthand for the corresponding element of $\mathfrak{A}$, as we have not assumed that the C^*-algebra in question is unital. Since the sequence $\{\|B^{1/2} - B^{1/2}e_\lambda\|\}_{\lambda\in\Lambda}$ converges to 0, it follows that

$$C^{1/2} = \lim_\lambda C^{1/2}e_\lambda,$$

and hence $e_\lambda C e_\lambda \to C$. As $e_\lambda \in \mathfrak{B}_+$, the above claim shows that $e_\lambda C e_\lambda \in \mathfrak{B}$, and thus $C \in \mathfrak{B}$. This proves that $\mathfrak{B}$ is indeed hereditary.

Now let $(u_\gamma)_{\gamma\in\Gamma}$ be an approximate unit for $\mathfrak{A}$. To see that A belongs to $\mathfrak{B}$, note that

$$A^2 = \lim_\gamma (Au_\gamma A) \in \mathfrak{B}$$

as $Au_\gamma A \in A\mathfrak{A}A$ for all γ. It is then immediate that $A = (A^2)^{1/2} \in \mathfrak{B}$.

■

The following useful result describes an important correspondence between hereditary C^*-subalgebras and closed left ideals.

Theorem 3.1.4. *Let $\mathfrak{A}$ be a C^*-algebra.*

(1) *If $\mathcal{L}$ is closed left ideal in $\mathfrak{A}$, then $\mathcal{L} \cap \mathcal{L}^*$ is a hereditary C^*-subalgebra of $\mathfrak{A}$.*

(2) *If $\mathcal{L}_1, \mathcal{L}_2$ are closed left ideal of $\mathfrak{A}$, then $\mathcal{L}_1 \subseteq \mathcal{L}_2$ if and only if $\mathcal{L}_1 \cap \mathcal{L}_1^* \subseteq \mathcal{L}_2 \cap \mathcal{L}_2^*$.*

(3) *If $\mathfrak{B}$ is a hereditary C^*-subalgebra of $\mathfrak{A}$, then*

$$\mathcal{L} := \{A \in \mathfrak{A} : A^*A \in \mathfrak{B}\}$$

defines a left ideal of $\mathfrak{A}$ and $\mathfrak{B} = \mathcal{L} \cap \mathcal{L}^*$.

Hence, the map $\mathcal{L} \mapsto \mathcal{L} \cap \mathcal{L}^*$ defines an inclusion preserving bijection between the left ideals of $\mathfrak{A}$ and the hereditary C^*-subalgebras of $\mathfrak{A}$.

Proof. (1) It is routine to verify that $\mathfrak{B} := \mathcal{L} \cap \mathcal{L}^*$ is a C^*-subalgebra of $\mathfrak{A}$ whenever $\mathcal{L}$ is a closed left ideal of $\mathfrak{A}$. To verify that $\mathfrak{B}$ is hereditary, let $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ be such that $0 \leq A \leq B$. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathfrak{B}$. We have that

$$0 \leq (I-e_\lambda)A(1-e_\lambda) \leq (I-e_\lambda)B(I-e_\lambda),$$

and hence the C^*-identity gives

$$\begin{aligned}\|A^{1/2} - A^{1/2}e_\lambda\|^2 &= \|(I-e_\lambda)A(I-e_\lambda)\| \\ &\leq \|(I-e_\lambda)B(I-e_\lambda)\| \\ &= \|B^{1/2} - B^{1/2}e_\lambda\|^2.\end{aligned}$$

Since the final term tends to 0, the above shows that $A^{1/2} = \lim_\lambda A^{1/2}e_\lambda$, and since each e_λ belongs to the left ideal $\mathcal{L}$, it follows that so too does $A^{1/2}$. This proves that $A^{1/2}$ (and hence A) is an element of $\mathfrak{B}$, and so $\mathfrak{B}$ is indeed hereditary.

(2) Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed left ideals of $\mathfrak{A}$. It is obvious that $\mathcal{L}_1 \cap \mathcal{L}_1^* \subseteq \mathcal{L}_2 \cap \mathcal{L}_2^*$ whenever $\mathcal{L}_1 \subseteq \mathcal{L}_2$. To see the converse, suppose that $\mathcal{L}_1 \cap \mathcal{L}_1^* \subseteq \mathcal{L}_2 \cap \mathcal{L}_2^*$ and let $A \in \mathcal{L}_1$. Once again, let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathcal{L}_1 \cap \mathcal{L}_1^*$, and note that since $A^*A \in \mathcal{L}_1$, we obtain

$$\lim_\lambda \|A - Ae_\lambda\|^2 = \lim_\lambda \|(I - e_\lambda)A^*A(I - e_\lambda)\| \leq \lim_\lambda \|A^*A(I - e_\lambda)\| = 0.$$

However, $e_\lambda \in \mathcal{L}_1 \cap \mathcal{L}_1^* \subseteq \mathcal{L}_2$, and hence the above implies that $A \in L_2$ as well. Therefore, $\mathcal{L}_1 \subseteq \mathcal{L}_2$.

(3) To see that L is indeed a left ideal of $\mathfrak{A}$, let $A, B \in \mathcal{L}$ and let $X \in \mathfrak{A}$. Then

$$(A + B)^*(A + B) \leq (A + B)^*(A + B) + (A - B)^*(A - B) = 2A^*A + 2B^*B \in \mathfrak{B},$$

and

$$(XA)^*(XA) = A^*X^*XA \leq \|X\|^2 A^*A \in \mathfrak{B}.$$

By heredity, $(A+B)^*(A+B)$ and $(XA)^*(XA)$ belong to $\mathfrak{B}$, and hence $A+B$ and XA belong to $\mathcal{L}$. A similar calculation shows that $\mathcal{L}$ is closed under scalar multiplication, and thus $\mathcal{L}$ is a left ideal of $\mathfrak{A}$. Since $\mathfrak{B}$ is closed in $\mathfrak{A}$, so too is $\mathcal{L}$.

If $B \in \mathfrak{B}$, then $B^*B \in \mathfrak{B}$ as well, so $B \in \mathcal{L}$. Thus, $\mathfrak{B} \subseteq \mathcal{L} \cap \mathcal{L}^*$. Conversely, let $0 \leq B \in \mathcal{L}$. It is evident that $B^2 \in \mathfrak{B}$, and hence $B = (B^2)^{1/2} \in \mathfrak{B}$. This shows that $\mathcal{L}_+ = (\mathcal{L} \cap \mathcal{L}^*)_+ \subseteq \mathfrak{B}_+$, and since every element of a C^*-algebra can be written as a linear combination of positive elements, it follows that $\mathcal{L} \cap \mathcal{L}^* \subseteq \mathfrak{B}$. Thus, $\mathfrak{B} = \mathcal{L} \cap \mathcal{L}^*$, as required.

The conclusions of (1), (2), and (3) demonstrate that the map $\mathcal{L} \mapsto \mathcal{L} \cap \mathcal{L}^*$ is well-defined, it is an inclusion preserving injection, and it is surjective, respectively. Hence, this map is an inclusion preserving bijection, as claimed.

■

With the above result in hand, a useful characterization of hereditary C^*-subalgebras is now readily obtained. The remainder of the section will be devoted to the following theorem and its several important consequences.

Theorem 3.1.5. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. Then $\mathfrak{B}$ is hereditary if and only if $BAB' \in \mathfrak{B}$ for all $A \in \mathfrak{A}$ and all $B, B' \in \mathfrak{B}$.*

Proof. Suppose first that $\mathfrak{B}$ is a hereditary C^*-subalgebra of $\mathfrak{A}$, let $B, B' \in \mathfrak{B}$, and let $A \in \mathfrak{A}$. By Theorem 3.1.4, there exists a left ideal $\mathcal{L}$ of $\mathfrak{A}$ such that $\mathfrak{B} = \mathcal{L} \cap \mathcal{L}^*$. Since $B', B^* \in \mathcal{L}$, we have that $(BA)B' \in \mathcal{L}$ and $(B'^*A^*)B^* \in \mathcal{L}$, and hence $BAB' \in \mathcal{L} \cap \mathcal{L}^* = \mathfrak{B}$.

Now suppose $\mathfrak{B}$ is a C^*-subalgebra of $\mathfrak{A}$ satisfying above property. Let $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ be such that $0 \leq A \leq B$. If $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $\mathfrak{B}$, we have that for all λ,

$$0 \leq (I - e_\lambda)A(I - e_\lambda) \leq (I - e_\lambda)B(I - e_\lambda),$$

and hence the C^*-identity implies (in the same way as before) that $A^{1/2} = \lim_\lambda A^{1/2}e_\lambda$. This, in turn, shows that

$$A = (A^{1/2})^*A^{1/2} = \lim_\lambda (A^{1/2}e_\lambda)^*(A^{1/2}e_\lambda) = \lim_\lambda e_\lambda A e_\lambda.$$

Since $e_\lambda A e_\lambda \in \mathfrak{B}$ for all $\lambda \in \Lambda$ by assumption, we conclude that $A \in \mathfrak{B}$, and hence $\mathfrak{B}$ is hereditary.

■

Theorem 3.1.5 is useful in proving the heredity of many C^*-subalgebras. In particular, if $\mathcal{J}$ is a closed ideal of a C^*-algebra $\mathfrak{A}$, then clearly we have that $\mathcal{J}\mathfrak{A}\mathcal{J} \subseteq \mathcal{J}$, and hence the following corollary is immediate. First recall that if $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are ideals in a C^*-algebra $\mathfrak{A}$, then we define $\mathcal{I}_1 \cdots \mathcal{I}_n$ to be the closed linear span of all products $A_1 \cdots A_n$, where $A_j \in \mathcal{I}_j$ for each $j \in \{1, \ldots, n\}$.

Corollary 3.1.6. *Every closed ideal of a C^*-algebra is a hereditary C^*-subalgebra.*

One interesting property of hereditary C^*-subalgebras is their ability to preserve the simplicity of the C^*-algebra to which they belong. In order to prove this useful result, we must examine the connection between closed ideals of a C^*-algebra and the closed ideals of its hereditary C^*-subalgebras.

Theorem 3.1.7. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B}$ be a hereditary C^*-subalgebra of $\mathfrak{A}$. If $\mathcal{J}$ is a closed ideal of $\mathfrak{B}$, then there is a closed ideal $\mathcal{I}$ of $\mathfrak{A}$ such that $\mathcal{J} = \mathfrak{B} \cap \mathcal{I}$.*

Proof. Let $\mathcal{I} := \mathfrak{A}\mathcal{J}\mathfrak{A}$ and note that $\mathcal{I}$ is a closed ideal of $\mathfrak{A}$. We claim that

- $\mathcal{J} = \mathcal{J}^3$ and
- $\mathfrak{B} \cap \mathcal{I} = \mathfrak{B}\mathcal{I}\mathfrak{B}$.

To see that the first assertion holds, note that clearly $\mathcal{J}^3 \subseteq \mathcal{J}$. Further, if $(e_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for $\mathcal{J}$ and $X \in \mathcal{J}$, then each $e_\lambda X e_\lambda \in \mathcal{J}^3$, and since

$$X = \lim_\lambda e_\lambda X e_\lambda$$

it follows that $X \in \mathcal{J}^3$ as well. For the second claim, note that since $\mathcal{I}$ is an ideal of $\mathfrak{A}$, it is obvious that $\mathfrak{B}\mathcal{I}\mathfrak{B} \subseteq \mathcal{I}$. Also, since

$$\mathfrak{B}\mathcal{I}\mathfrak{B} = \mathfrak{B}(\mathfrak{A}\mathcal{J}\mathfrak{A})\mathfrak{B} \subseteq \mathfrak{B}(\mathfrak{A}\mathfrak{B}\mathfrak{A})\mathfrak{B} = (\mathfrak{B}\mathfrak{A}\mathfrak{B})\mathfrak{B}(\mathfrak{B}\mathfrak{A}\mathfrak{B}),$$

and the heredity of $\mathfrak{B}$ in $\mathfrak{A}$ implies that $\mathfrak{B}\mathfrak{A}\mathfrak{B} \subseteq \mathfrak{B}$ by Theorem 3.1.5, it is easy to see that

$$\mathfrak{B}\mathcal{I}\mathfrak{B} \subseteq \mathfrak{B}^3 \subseteq \mathfrak{B}.$$

Combining the above observations, we deduce that $\mathfrak{B}\mathcal{I}\mathfrak{B} \subseteq \mathfrak{B} \cap \mathcal{I}$. Finally, if $X \in \mathfrak{B} \cap \mathcal{I}$ and $(u_\gamma)_{\gamma \in \Gamma}$ is an approximate unit for $\mathfrak{B}$, then clearly $u_\gamma X u_\gamma \in \mathfrak{B}\mathcal{I}\mathfrak{B}$ for all γ, and hence

$$X = \lim_\gamma u_\gamma X u_\gamma$$

must also belong to $\mathfrak{B}\mathcal{I}\mathfrak{B}$.

With the above results in mind, we note that

$$\mathfrak{B} \cap \mathcal{I} = \mathfrak{B}\mathcal{I}\mathfrak{B} = \mathfrak{B}(\mathfrak{A}\mathcal{J}^3\mathfrak{A})\mathfrak{B} = (\mathfrak{B}\mathfrak{A}\mathcal{J})\mathcal{J}(\mathcal{J}\mathfrak{A}\mathfrak{B}),$$

and since $\mathfrak{B}\mathfrak{A}\mathcal{J}$ and $\mathcal{J}\mathfrak{A}\mathfrak{B}$ are both contained in $\mathfrak{B}$ by Theorem 3.1.5, it follows that

$$\mathfrak{B} \cap \mathcal{I} \subseteq \mathfrak{B}\mathcal{J}\mathfrak{B} = \mathcal{J}.$$

We conclude by noting that since $\mathcal{J} = \mathcal{J}^3 \subseteq \mathfrak{A}\mathcal{J}\mathfrak{A} = \mathcal{I}$ (and of course $\mathcal{J} \subseteq \mathfrak{B}$), it is clear that $\mathcal{J} = \mathfrak{B} \cap \mathcal{I}$ and hence the proof is complete.

■

Corollary 3.1.8. *Every hereditary C^*-subalgebra of a simple C^*-algebra is simple.*

Proof. If $\mathfrak{B}$ is a hereditary C^*-subalgebra of a simple C^*-algebra $\mathfrak{A}$ and $\mathcal{J}$ is an ideal of $\mathfrak{B}$, then Theorem 3.1.7 indicates that $\mathcal{J} = \mathfrak{B} \cap \mathcal{I}$ for some ideal $\mathcal{I}$ of $\mathfrak{A}$. By simplicity of $\mathfrak{A}$, either $\mathcal{I} = \{0\}$ or $\mathcal{I} = \mathfrak{A}$, and hence either $\mathcal{J} = \{0\}$ or $\mathcal{J} = \mathfrak{B}$. This completes the proof.

■

3.2 Infinite Projections and Purely Infinite C^*-Algebras

In order to discuss the concept of a purely infinite C^*-algebra, we must first define what it means for a projection in a C^*-algebra to be *infinite*. Recall that two projections, P and Q, in a C^*-algebra $\mathfrak{A}$ are said to be **Murray–von Neumann equivalent** (written $P \sim_0 Q$) if there exists a partial isometry $S \in \mathfrak{A}$ such that $P = S^*S$ and $Q = SS^*$. In this case, P is called the **source projection** of S, and Q is called the **range projection** of S.

Definition 3.2.1. *Let $\mathfrak{A}$ be a C^*-algebra and $P \in \mathfrak{A}$ be a projection. We say P is*

- ***infinite** if P is Murray–von Neumann equivalent to a proper subprojection of itself.*
- ***properly infinite** if there are orthogonal projections $Q_1, Q_2 \in \mathfrak{A}$ such that $Q_1 \sim_0 Q_2 \sim_0 P$, and $Q_1 + Q_2 \leq P$.*

*The C^*-algebra $\mathfrak{A}$ is called **infinite** if it contains an infinite projection, and **properly infinite** if it contains a properly infinite projection.*

Definition 3.2.2. *A C^*-algebra $\mathfrak{A}$ is called **purely infinite** if every non-zero hereditary C^*-subalgebra of $\mathfrak{A}$ contains an infinite projection.*

The Calkin algebra, $\mathcal{Q}(\mathcal{H})$, where $\mathcal{H}$ is a complex, separable, infinite-dimensional Hilbert space is an example of a purely infinite C^*-algebra, as it satisfies a certain algebraic characterization of this property that is outlined in Theorem 3.2.11. Likewise, the Cuntz algebra, $\mathcal{O}_n$ is purely infinite by the same reasoning [9, Theorem V.4.6]. For now we will examine some familiar C^*-algebras that are *not* purely infinite.

Example 3.2.3. *If $\mathcal{H}$ is a complex, separable, infinite-dimensional Hilbert space, then $\mathcal{B}(\mathcal{H})$ is not purely infinite. To see this, note that $\mathcal{K}(H)$ is an ideal of $\mathcal{B}(\mathcal{H})$ and hence a hereditary C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ by Corollary 3.1.6 . The fact that $\mathcal{B}(\mathcal{H})$ is not purely infinite now follows as $\mathcal{K}(\mathcal{H})$ does not contain infinite projections.*

Example 3.2.4. *If $\mathfrak{A}$ is an abelian C^*-algebra then $\mathfrak{A}$ is not purely infinite. To see this, note that if $P \in \mathfrak{A}$ is a projection and Q is a projection in $\mathcal{A}$ with $P \sim_0 Q$, then there exists a partial isometry $S \in \mathfrak{A}$ with $SS^* = P$ and $S^*S = Q$. Since $\mathfrak{A}$ is abelian, it is immediate that $P = Q$, and hence Q cannot be a proper subprojection of P. Hence, no projection in $\mathfrak{A}$ is infinite and the result holds.*

As a consequence to the above, if X is a compact Hausdorff space, then the space $\mathcal{C}(X)$ of continuous, complex-valued functions on X is never purely infinite.

Some basic properties of purely infinite C^*-algebras are summarized in the following proposition. These facts lead to an important corollary, which states that *every non-zero projection* in a purely infinite C^*-algebra is an infinite projection.

Proposition 3.2.5. *Let $\mathfrak{A}$ be a purely infinite C^*-algebra.*

(1) *If $\mathfrak{B}$ is a hereditary C^*-subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ is also purely infinite.*

(2) *If $P \in \mathfrak{A}$ is an infinite projection and $Q \in \mathfrak{A}$ is a projection with $P \leq Q$, then Q is infinite.*

(3) *If $\mathfrak{A}$ is unital, then I is an infinite projection.*

Proof. (1) Let $\mathfrak{C}$ be a hereditary C^*-subalgebra of $\mathfrak{B}$. If we can show that $\mathfrak{C}$ is also a hereditary C^*-subalgebra of $\mathfrak{A}$, then $\mathfrak{C}$ will contain an infinite projection (by virtue of $\mathfrak{A}$ being purely infinite) and we may deduce that $\mathfrak{B}$ is purely infinite.

Suppose $A \in \mathfrak{A}$ and $C \in \mathfrak{C}$ are such that $0 \leq A \leq C$. Since $C \in \mathfrak{C} \subseteq \mathfrak{B}$, it follows that $A \in \mathfrak{B}$ by heredity of $\mathfrak{B}$ in $\mathfrak{A}$. Consequently, the heredity of $\mathfrak{C}$ in $\mathfrak{B}$ demonstrates that $A \in \mathfrak{C}$. Therefore, $\mathfrak{C}$ is a hereditary C^*-subalgebra of $\mathfrak{A}$, and the above remarks imply that $\mathfrak{C}$ is purely infinite.

(2) Let $V \in \mathfrak{A}$ be such that $VV^* = P$ and $R := V^*V < P$. Since $PVR = (VV^*)V(V^*V) = V$, we see that

$$V(Q-P) = PVR(Q-P) = PVR - PVR = 0,$$

and

$$(Q-P)V = (Q-P)PVR = PVR - PVR = 0.$$

If we define $S := V + Q - P$, then clearly

$$SS^* = (V+Q-P)(V^*+Q-P) = VV^* + Q - P = Q,$$

and

$$S^*S = (V^*+Q-P)(V+Q-P) = V^*V + Q - P = R + Q - P =: T$$

is a proper subprojection of Q. Hence $Q \sim_0 T < Q$, and we conclude that Q is an infinite projection.

(3) If P is an infinite projection in $\mathfrak{A}$, then $P \leq I$. The second statement of this proposition now implies that I is infinite.

■

Corollary 3.2.6. *Let $\mathfrak{A}$ be a purely infinite C^*-algebra. Then every non-zero projection in $\mathfrak{A}$ is infinite.*

Proof. Let P be a non-zero projection in $\mathfrak{A}$. By Propositions 3.1.2 and 3.2.5, it follows that the corner $P\mathfrak{A}P$ is a purely infinite C^*-algebra. Further, $P\mathfrak{A}P$ is unital with identity P. By Proposition 3.2.5, we conclude that P is an infinite projection.

■

One of the main results of this section is Theorem 3.2.8, which states that every infinite projection in a simple C^*-algebra is properly infinite in a powerful way. This fact is essential in extending the results of Section 2 to the setting of purely infinite C^*-algebras, and hence will be applied heavily throughout the coming sections. We prove the following lemma before tackling this important result.

Lemma 3.2.7. *Let $\mathfrak{A}$ be a simple C^*-algebra and Q be a projection in $\mathfrak{A}$. If $0 \neq P \in \mathfrak{A}_+$, then there exist elements $Z_1, \ldots, Z_n \in \mathfrak{A}$ such that $Q = \sum_{i=1}^n Z_i^* P Z_i$.*

Proof. Since $P \neq 0$ we may assume that $\|P\| = 1$. Since $\mathfrak{A}$ is simple, we have that $\overline{\langle P \rangle} = \mathfrak{A}$. This implies that $Q \in \overline{\langle P \rangle}$, and so there exist $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathfrak{A}$ with $\|Q - \sum_{i=1}^n X_i P Y_i\| < 1/2$.

Notice that since $P > 0$, $P = B^*B$ for some $B \in \mathfrak{A}$. Hence

$$\begin{aligned} 0 &\leq (B(X_i^* - Y_i))^*(B(X_i^* - Y_i)) \\ &= (X_i - Y_i^*)B^*B(X_i^* - Y_i) \\ &= X_i P X_i^* - X_i P Y_i - Y_i^* P X_i^* + Y_i^* P Y_i. \end{aligned}$$

In particular, this implies that

$$X_iPY_i + Y_i^*PX_i^* \le X_iPX_i^* + Y_i^*PY_i.$$

Passing to the unitization of $\mathfrak{A}$, one sees that

$$\begin{aligned} 2Q - \sum_{i=1}^n X_iPY_i - \sum_{i=1}^n Y_i^*PX_i^* &\le \left\| 2Q - \sum_{i=1}^n X_iPY_i - \sum_{i=1}^n Y_i^*PX_i^* \right\| I \\ &\le \left(\left\| Q - \sum_{i=1}^n X_iPY_i \right\| + \left\| \left(Q - \sum_{i=1}^n X_iPY_i \right)^* \right\| \right) I \le I. \end{aligned}$$

By conjugating both sides by Q (and thus returning to $\mathfrak{A}$), we see that

$$2Q - \sum_{i=1}^n QX_iPY_iQ - \sum_{i=1}^n QY_i^*PX_i^*Q \le Q,$$

and upon rearrangement it follows that

$$Q \le \sum_{i=1}^n QX_iPY_iQ + \sum_{i=1}^n QY_i^*PX_i^*Q \le \sum_{i=1}^n QX_iPX_i^*Q + \sum_{i=1}^n QY_i^*PY_iQ =: A \le cQ,$$

where $c := \sum_{i=1}^n (\|X_i\|^2 + \|Y_i\|^2)$. With respect to the decomposition $Q\mathcal{H} \oplus (Q\mathcal{H})^\perp$, we may write

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Note that since $Q \le A \le cQ$, one has

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \le \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \le \begin{bmatrix} cI & 0 \\ 0 & 0 \end{bmatrix}.$$

One can now deduce that $A_2 = A_3 = A_4 = 0$ and $\sigma(A_1) \subseteq [1, c]$, thereby showing $\sigma(A) \subseteq \{0\} \cup [1, c]$. Define the function f on $\sigma(A)$ by

$$f(x) = \left\{ \begin{array}{ll} 0, & \text{if } x = 0 \\ x^{-1/2}, & \text{if } x \in [1, c] \end{array} \right. ,$$

and note that

$$f(A)Af(A) = \begin{bmatrix} f(A_1) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f(A_1) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1/2} A_1 A_1^{-1/2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = Q.$$

Since $f(A)$ is self-adjoint, we may conclude that

$$\begin{aligned} Q &= f(A) \left(\sum_{i=1}^n QX_iPX_i^*Q + \sum_{i=1}^n QY_i^*PY_iQ \right) f(A) \\ &= \sum_{i=1}^n (X_i^*Qf(A))^*P(X_i^*Qf(A)) + \sum_{i=1}^n (Y_iQf(A))^*P(Y_iQf(A)), \end{aligned}$$

and the proof is complete.

■

Theorem 3.2.8. *Let $\mathfrak{A}$ be a simple, infinite C^*-algebra and $Q \in \mathfrak{A}$ be an infinite projection. Then there exist partial isometries $\{T_k\}_{k=0}^{\infty} \subseteq \mathfrak{A}$ with pairwise orthogonal range projections such that $Q = T_k^* T_k$ and $\sum_{k=1}^{N} T_k T_k^* < Q$ for all $N \geq 1$.*

Proof. Let S be a partial isometry in $\mathfrak{A}$ such that $P := SS^* < S^*S = Q$. By working in the corner $\mathfrak{B} := Q\mathfrak{A}Q$, we have that $\mathfrak{B}$ is a unital hereditary C^*-subalgebra of $\mathfrak{A}$ with unit given by $I = Q$. Since $\mathfrak{A}$ was assumed to be simple, Corollary 3.1.8 implies that $\mathfrak{B}$ is simple as well. Moreover, $P = QPQ \in \mathfrak{B}$ and

$$S = SS^*S = PS = QPSS^*S = Q(PS)Q \in \mathfrak{B}.$$

Hence, Q is an infinite projection in $\mathfrak{B}$ and so we may apply Lemma 3.2.7 to find elements $X_1, \ldots, X_n \in \mathfrak{B}$ that satisfy

$$\sum_{i=1}^{n} X_i^*(I - P)X_i = I,$$

as $I - P \neq 0$.

We claim that $\{S^i(I-P) : i \geq 1\}$ is a collection of partial isometries, each with initial projection $I - P$, and such that $(S^i(I-P))^* S^j(I-P) = 0$ whenever $i \neq j$. Indeed,

$$(S^i(I-P))^* S^i(I-P) = (I-P)(S^*)^i S^i(I-P) = (I-P)I(I-P) = (I-P).$$

Further, when $j < i$, we have

$$\begin{aligned}(S^i(I-P))^* S^j(I-P) &= (I-P)(S^*)^i S^j(I-P) \\ &= (I-P)S^{j-i}(I-P) \\ &= \underbrace{(S^{j-i} - PS^{j-i})}_{=0}(I-P) = 0.\end{aligned}$$

We obtain a similar result for the case when $i < j$.

Now define $T_1 := \sum_{i=1}^{n} S^{i-1}(I-P)X_i$ and note that

$$T_1^* T_1 = \sum_{i,j} X_i^*(I-P)(S^*)^{i-1}S^{j-1}(I-P)X_j = \sum_i X_i^*(I-P)X_i = I,$$

so T_1 is a partial isometry. For $k \geq 2$, we define $T_k := S^{n(k-1)}T_1$ and note that

$$T_k^* T_k = T_1^*(S^*)^{n(k-1)}S^{n(k-1)}T_1 = T_1^* T_1 = I.$$

If $\ell > k$, then $\ell - k \geq 1$, and hence for each $i \in \{1, \ldots, n\}$, we have

$$\alpha_i := n(\ell - k) - (i-1) \geq n(\ell - k) - (n-1) = n(\ell - k - 1) + 1 > 0.$$

This implies that

$$\begin{aligned}T_k^* T_\ell &= T_1^*(S^*)^{n(k-1)}S^{n(\ell-1)}T_1 \\ &= \left(\sum_{i=1}^{n} X_i^*(I-P)(S^*)^{i-1}\right) S^{n(\ell-k)}T_1 \\ &= \sum_{i=1}^{n} X_i^*(I-P)S^{\alpha_i}T_1 \\ &= \sum_{i=1}^{n} X_i^* \underbrace{(S^{\alpha_i} - PS^{\alpha_i})}_{=0} T_1 = 0.\end{aligned}$$

It follows that $\{T_kT_k^* : k \in \mathbb{N}\}$ is a family of pairwise orthogonal projections with $T_k^*T_k = I$ for all k. All that remains to show is that $\sum_{i=1}^N T_kT_k^* < I$ for all $N \geq 1$.

First notice that

$$\begin{aligned} T_1T_1^*\left(\sum_j S^{j-1}(I-P)(S^*)^{j-1}\right) &= T_1\left(\sum_i X_i^*(I-P)(S^*)^{i-1}\right)\left(\sum_j S^{j-1}(I-P)(S^*)^{j-1}\right) \\ &= T_1\sum_{i,j} X_i^*(I-P)(S^*)^{i-1}S^{j-1}(I-P)(S^*)^{j-1} \\ &= T_1\sum_i X_i^*(I-P)(S^*)^{i-1} = T_1T_1^*, \end{aligned}$$

and similarly $T_1T_1^* = \left(\sum_j S^{j-1}(I-P)(S^*)^{j-1}\right)T_1T_1^*$. Since $T_1T_1^* \leq I$, it follows that

$$\begin{aligned} T_1T_1^* &\leq \sum_{i,j} S^{i-1}(I-P)(S^*)^{i-1}S^{j-1}(I-P)(S^*)^{j-1} \\ &= \sum_i S^{i-1}(I-P)(S^*)^{i-1} \\ &= \sum_i S^{i-1}(S^*)^{i-1} - \sum_i S^{i-1}P(S^*)^{i-1} \\ &= \sum_i S^{i-1}(S^*)^{i-1} - \sum_i S^i(S^*)^i = I - S^n(S^*)^n. \end{aligned}$$

Thus,

$$T_kT_k^* = S^{n(k-1)}T_1T_1^*(S^*)^{n(k-1)} \leq S^{n(k-1)}(I-S^n(S^*)^n)(S^*)^{n(k-1)} = S^{n(k-1)}(S^*)^{n(k-1)} - S^{nk}(S^*)^{nk},$$

and we may conclude that for $N \geq 1$,

$$\sum_{k=1}^N T_kT_k^* \leq \sum_{k=1}^N (S^{n(k-1)}(S^*)^{n(k-1)} - S^{nk}(S^*)^{nk}) = I - S^{nN}(S^*)^{nN} < I$$

This completes the proof.

■

Corollary 3.2.9. *Every simple, infinite C^*-algebra $\mathfrak{A}$ is properly infinite.*

Proof. Let Q be any infinite projection in $\mathfrak{A}$. Simply take $N = 2$ in Theorem 3.2.8 to obtain two orthogonal projections, each equivalent to Q, and whose sum is a proper subprojection of Q.

■

The following corollary illustrates yet another interesting property of infinite projections, which will see an abundance of applications in the approximation results to follow.

Corollary 3.2.10. *If P and Q are projections in a simple C^*-algebra, $\mathfrak{A}$, and P is infinite, then Q is equivalent to a subprojection of P.*

Proof. By Lemma 3.2.7, there exist $X_1, \ldots, X_n \in \mathfrak{A}$ such that $Q = \sum_{i=1}^n X_i P X_i^*$. Further, an application of Theorem 3.2.8 yields partial isometries $S_1, \ldots, S_n \in \mathfrak{A}$ that satisfy $P = S_i^* S_i$ and $\sum_{i=1}^n S_i S_i^* < P$. As seen in the proof of Theorem 3.2.8, we may arrange that $S_i^* S_j = 0$ whenever $i \neq j$. Define $S := \sum_{i=1}^n X_i P S_i^*$ and note that since

$$SS^* = \sum_{i,j} X_i P S_i^* S_j P X_j^* = \sum_i X_i P X_i^* = Q,$$

we have that S is a partial isometry with initial projection equivalent to Q. If we can show that the initial projection of S is a subprojection of P, then the proof will be complete.

For each $i \in \{1, \ldots, n\}$, we have

$$PS_i P = PS_i S_i^* S_i = S_i S_i^* S_i = S_i P,$$

as $S_i S_i^* \leq \sum_i S_i S_i^* < P$. Thus,

$$PS^* S = \sum_{i,j} PS_i P X_i^* X_j P S_j^* = \sum_{i,j} S_i P X_i^* X_j P S_j^* = S^* S,$$

and by considering adjoints, we obtain $S^* SP = S^* S$. This proves that $S^* S \leq P$, as desired. ∎

We end this subsection by presenting an algebraic characterization for purely infinite C^*-algebras. This, in particular, will be used to show that the Calkin algebra is purely infinite. Since the closure of nilpotent operators in this algebra was fully characterized in Theorem 2.3.4, the fact that $\mathcal{Q}(\mathcal{H})$ is a simple, purely infinite C^*-algebra now motivates our ambition for extending these characterizations to other simple, purely infinite C^*-algebras.

Theorem 3.2.11. *Let $\mathfrak{A}$ be a simple, unital C^*-algebra of dimensional at least 2. The following are equivalent:*

(1) *$\mathfrak{A}$ is purely infinite.*

(2) *for all non-zero $A \in \mathfrak{A}$, there exist $X, Y \in \mathfrak{A}$ such that $XAY = I$.*

(3) *for all non-zero $A \in \mathfrak{A}_+$ and $\varepsilon > 0$, there exists $X \in \mathfrak{A}$ with $\|X\| < \|A\|^{-1/2} + \varepsilon$ and $XAX^* = I$.*

Proof. Firstly, if (3) holds and $A \in \mathfrak{A}$ is non-zero, then $A^*A \geq 0$, and by assumption there exists $X \in \mathfrak{A}$ with $XA^*AX = I$, from which we obtain (2).

Now assume (2) is true. To show that (1) holds, we will make use of the fact that (3) is satisfied without the norm estimate. Let $A \in \mathfrak{A}_+$ be non-zero. Applying the assumptions of (2) to $A^{1/2}$, there exist $X, Y \in \mathfrak{A}$ with $XA^{1/2}Y = I$. Hence, we obtain

$$I = I \cdot I^* = XA^{1/2}YY^*A^{1/2}X^* \leq \|Y\|^2 XAX^*.$$

This implies that $Z := XAX^*$ belongs to $\mathrm{GL}(\mathfrak{A}_+)$, and hence

$$(Z^{-1/2}X)A(Z^{-1/2}X)^* = I.$$

This is exactly the result of (3) without the norm estimate, as claimed.

Now let $\mathfrak{B}$ be a hereditary C^*-subalgebra of $\mathfrak{A}$, and let B be a non-zero, non-invertible positive element of $\mathfrak{B}$ (which exists by the heredity of $\mathfrak{B}$ together with the assumption that $\mathfrak{A}$ has dimension at least 2). The above analysis implies the existence of an element $W \in \mathfrak{A}$ such that $WBW^* = I$. Define $S := B^{1/2}W^*$ and note that $S^*S = WBW^* = I$. If S were invertible, then we would have

$$B^{1/2}W^*WB^{1/2} = SS^* = I.$$

In this case, $0 \notin \sigma(B^{1/2})$, and hence the spectral mapping theorem implies that $0 \notin \sigma(B)$. This contradicts the assumption that B is non-invertible, and we thus have that S is a proper isometry.

Define $P := SS^*$ and notice that

$$P = B^{1/2}W^*WB^{1/2} \leq \|W\|^2 B.$$

By heredity of $\mathfrak{B}$, it follows that $P \in \mathfrak{B}$. Let us show that this P is indeed an infinite projection. If we define Q to be the projection SPS^* then it is clear that $Q = SPS^* \leq SS^* = P$, as $P \leq I$. Moreover, $P = (SP)^*(SP)$ and $Q = (SP)(SP)^*$, so $Q \sim_0 P$. One must only verify that Q is a *proper* subprojection of P. Suppose to the contrary that $Q = P$. Then

$$S^*PS = S^*QS = S^*(SPS^*)S = (S^*S)Q(S^*S) = Q = SPS^*$$

from which it follows that $S(SS^*S^*) = S^*(SS^*)S = I$. This shows that S has a right inverse (as well as a left inverse as S is an isometry) and therefore S is invertible. This contradiction demonstrates that Q must indeed be a proper subprojection of P, and we conclude that since $\mathfrak{B}$ contains an infinite projection, $\mathfrak{A}$ is purely infinite. This proves (1).

Finally, suppose that (1) holds. Let $A \in \mathfrak{A}_+$ with $\|A\| = 1$. Let $\varepsilon \in (0, 1/2)$ and define the continuous function f on $[0, 1]$ by

$$f(t) = \begin{cases} 0, & 0 \leq t \leq 1-\varepsilon \\ 1 - \varepsilon^{-1}(1-t), & 1-\varepsilon \leq t \leq 1 \end{cases}.$$

Consider the hereditary C^*-subalgebra $\mathfrak{B} := \overline{f(A)\mathfrak{A}f(A)}$ and note that by assumption, $\mathfrak{B}$ contains an infinite projection, P. Lemma 3.2.10 implies that I is equivalent to a subprojection of P, so there is an isometry S whose range projection satisfies $SS^* \leq P$. Note that $P(SS^*) = SS^*$, and hence right multiplication by S yields $PS = S$. Since $\sigma(A) \subseteq [0, 1]$, we have that $f(A)$, and hence every element of $\mathfrak{B}$ is of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}$$

with respect to the decomposition $E_A([0, 1-\varepsilon))\mathcal{H} \oplus E_A([1-\varepsilon, 1])\mathcal{H}$, where E_A denotes the spectral measure corresponding to A. In particular, P has the above form where $Z = Q$ is a projection. With this in mind, one can see that

$$P = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = E_A([1-\varepsilon, 1]) \leq (1-\varepsilon)^{-1}A,$$

and so it follows that $PAP \geq (1-\varepsilon)P$. As a result,

$$B := S^*AS = S^*PAPS \geq (1-\varepsilon)S^*PS = (1-\varepsilon)I,$$

so $B \in \mathrm{GL}(\mathfrak{A}_+)$, and thus $(B^{-1/2}S^*)A(SB^{-1/2}) = I$. Finally,

$$\|SB^{-1/2}\| \leq \|S\|\|B^{-1/2}\| \leq (1-\varepsilon)^{-1/2} < 1 + \varepsilon$$

for $\varepsilon \in (0, 1/2)$. This establishes (3) and the proof is complete.

∎

Corollary 3.2.12. *Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space. The Calkin algebra $\mathcal{Q}(\mathcal{H})$ is purely infinite and simple.*

Proof. It is well known that $\mathcal{K}(\mathcal{H})$ is the only non-trivial closed ideal of $\mathcal{B}(\mathcal{H})$ whenever $\mathcal{H}$ is separable and infinite-dimensional. By correspondence, it is clear that $\mathcal{Q}(\mathcal{H})$ is simple.

Let $T \in \mathcal{B}(\mathcal{H})$ be non-compact. Our goal is to appeal to Theorem 3.2.11 by finding elements $A, B \in \mathcal{B}(\mathcal{H})$ such that $\pi(I) = \pi(A)\pi(T)\pi(B)$, where $\pi : \mathcal{B}(\mathcal{H})\mathcal{Q}(\mathcal{H})$ is the canonical quotient map. With this in mind, we may assume that $T \geq 0$ and $\|T\| = 1$ (by replacing T by $\frac{T^*T}{\|T^*T\|}$ if necessary). There exists a spectral measure $E_T : \mathrm{Bor}(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ such that $E_T(\Delta)$ is a projection for all Borel sets Δ on $\sigma(T)$, and

$$T = \int_0^1 x \, dE_T(x).$$

Since $T \notin \mathcal{K}(\mathcal{H})$, there exists $\varepsilon \in (0, 1]$ such that $E_T((\varepsilon, 1])$ is a projection of infinite rank. Define ι and f on $[0, 1]$ by $\iota(x) = x$ and

$$f(x) = \begin{cases} 0, & 0 \leq x \leq \varepsilon \\ 1/x, & \varepsilon < x \leq 1 \end{cases},$$

and note that the map $\varphi : L^\infty_{\mathrm{Bor}}(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ given by $\varphi(g) = \int_0^1 g(x)\, dE_T(x)$ is a $*$-homomorphism. Having said this, we observe that by defining $S := \varphi(f)$, we have

$$TS = \varphi(\iota)\varphi(f) = \varphi(\iota f) = \int_0^1 x f(x)\, dE_T(x) = \int_\varepsilon^1 dE_T(x) = E_T((\varepsilon, 1]).$$

Hence, by replacing T by TS if necessary, we may assume that T is a projection of infinite rank. In this case, T is equivalent to I, and so there is a partial isometry U such that $U^*U = T$ and $UU^* = I$. But then $I = UU^*UU^* = UTU^*$ and we see that

$$\pi(I) = \pi(U)\pi(T)\pi(U)^*.$$

By Theorem 3.2.11, we deduce that $\mathcal{Q}(\mathcal{H})$ is purely infinite.

■

3.3 C^*-Algebras of Real Rank Zero

To sum up Section 3, we will investigate the consequences that arise when considering the *real rank* of a unital, simple, purely infinite C^*-algebra. For convenience, we will adopt the notation $\mathfrak{A}_{sa}$ to describe the set of self-adjoint elements of a C^*-algebra $\mathfrak{A}$.

Definition 3.3.1. *Let $\mathfrak{A}$ be a unital C^*-algebra. We say $\mathfrak{A}$ has* ***real rank zero*** *if the invertible, self-adjoint elements of $\mathfrak{A}$ are dense in $\mathfrak{A}_{sa}$.*

As the following theorem states, every unital, simple, purely infinite C^*-algebra is of real rank zero. This fact, combined with Corollary 3.3.5, demonstrates that such C^*-algebras are incredibly saturated with projections.

Theorem 3.3.2. *If $\mathfrak{A}$ is a unital, simple, purely infinite C^*-algebra, then $\mathfrak{A}$ has real rank zero.*

Proof. Let $A \in \mathfrak{A}$ be self-adjoint and let $\varepsilon > 0$. Define $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$f_\varepsilon(t) = \begin{cases} 0, & |t| \leq \varepsilon \\ t - \varepsilon, & t \geq \varepsilon \\ t + \varepsilon, & t \leq -\varepsilon \end{cases},$$

and $g_\varepsilon(t) = \max\{\varepsilon - |t|, 0\}$. Let P be an infinite projection in the hereditary C^*-subalgebra

$$\mathfrak{B} := \overline{g_\varepsilon(A)\mathfrak{A}g_\varepsilon(A)}$$

of $\mathfrak{A}$. An application of Lemma 3.2.10 yields a partial isometry $S \in \mathfrak{A}$ such that $S^*S = I - P$ and $Q := SS^*$ is a subprojection of P. If E_A denotes the spectral measure corresponding to the element A, then by examining the matrix representations of P and $f_\varepsilon(A)$ associated to the decomposition

$$E_A((-\infty, \varepsilon))\mathcal{H} \oplus E_A([-\varepsilon, \varepsilon])\mathcal{H} \oplus E_A((\varepsilon, \infty))\mathcal{H},$$

one can verify that $f_\varepsilon(A) = (I - P)f_\varepsilon(A)(I - P)$. From this it follows that, with respect to the decomposition $(I - P)\mathcal{H} \oplus Q\mathcal{H} \oplus (P - Q)\mathcal{H}$, we have

$$B := f_\varepsilon(A) + \varepsilon(S + S^*) + \varepsilon(P - Q) \simeq \begin{bmatrix} f_\varepsilon(A) & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix},$$

where S acts as the matrix unit E_{21}. Since

$$B = \begin{bmatrix} f_\varepsilon(A) & \varepsilon \\ \varepsilon & 0 \end{bmatrix} \oplus \varepsilon$$

and the matrix on the left has inverse given by

$$\begin{bmatrix} 0 & \varepsilon^{-1} \\ \varepsilon^{-1} & -\varepsilon^2 f_\varepsilon(A) \end{bmatrix},$$

it is clear that B is invertible and self-adjoint. Moreover,

$$\|B - A\| \leq \underbrace{\|f_\varepsilon(A) - A\|}_{\leq \varepsilon} + \varepsilon \underbrace{\|S + S^* + (P - Q)\|}_{=1} \leq 2\varepsilon$$

and we conclude that the invertible, self-adjoint elements of $\mathfrak{A}$ form a dense subset of the self-adjoint elements in $\mathfrak{A}$. That is, $\mathfrak{A}$ has real rank zero.

■

The real rank zero property has several other equivalent formulations that are outlined in the following theorem. Since our interest is not in C^*-algebras of real rank zero themselves, but rather in simple, purely infinite C^*-algebras as a particular example, we refer the reader to a proof of this result in [9, Theorem V.7.3].

Theorem 3.3.3. *Let $\mathfrak{A}$ be a C^*-algebra. Then the following are equivalent:*

(1) *$\mathfrak{A}$ has real rank zero.*

(2) *The elements of $\mathfrak{A}_{sa}$ with finite spectrum are dense in $\mathfrak{A}_{sa}$.*

(3) *Every hereditary C^*-subalgebra of $\mathfrak{A}$ has an approximate unit consisting of projections.*

If $\mathfrak{A}$ is a C^*-algebra of real rank zero, then the (aptly-named) hereditary C^*-subalgebras of $\mathfrak{A}$ also inherit this property. This is illustrated in the following corollary:

Corollary 3.3.4. *If $\mathfrak{A}$ is a C^*-algebra of real rank zero and $\mathfrak{B}$ is a hereditary subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ has real rank zero.*

Proof. Let $\mathfrak{C}$ be a hereditary C^*-subalgebra of $\mathfrak{B}$, and note that the arguments from the proof of Proposition 3.2.5 imply that $\mathfrak{C}$ must also be a hereditary C^*-subalgebra of $\mathfrak{A}$, as seen in the proof of Proposition 3.2.5. Since $\mathfrak{A}$ has real rank zero, Theorem 3.3.3 demonstrates that $\mathfrak{C}$ has an approximate unit consisting of projections. Since this must be true of all hereditary C^*-subalgebras of $\mathfrak{B}$, we deduce that $\mathfrak{B}$ must have real rank zero by Theorem 3.3.3.

■

The usefulness of real rank zero C^*-algebras in understanding the closure of nilpotent operators can be seen in the following proposition, which states that the collection of $\mathbb{R}$-linear combinations of projections forms a dense subset within the set of self-adjoint elements.

Corollary 3.3.5. *Let $\mathfrak{A}$ be a C^*-algebra of real rank zero and $A \in \mathfrak{A}_{sa}$. For $\varepsilon > 0$, there exist pairwise orthogonal projections $P_1, \ldots, P_n \in \mathfrak{A}$, and scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that*

$$\|\alpha_1 P_1 + \cdots + \alpha_n P_n - A\| < \varepsilon.$$

Moreover, if A were in fact a positive element of $\mathfrak{A}$, then the scalars $\alpha_1, \ldots, \alpha_n$ may be chosen to be positive.

Proof. By Theorem 3.3.3, there is a exists an element $S \in \mathfrak{A}_{sa}$ with finite spectrum and such that $\|A - S\| \leq \varepsilon$. Let $\alpha_1, \ldots, \alpha_n$ denote the distinct elements of $\sigma(S) \subseteq \mathbb{R}$, and for each $i \in \{1, \ldots, n\}$, define P_i to be the spectral projection of S corresponding to $\{\alpha_i\}$. It follows that

$$S = \sum_{i=1}^{n} \alpha_i P_i,$$

and hence we obtain the required approximation.

If instead A were in $\mathfrak{A}_+$, the same arguments as in the previous case give rise to an element $S = \alpha_1 P_1 + \cdots + \alpha_n P_n$, where $\sigma(S) = \{\alpha_1, \ldots, \alpha_n\}$, P_i is the spectral projection of S corresponding to $\{\alpha_i\}$, and $\|A - S\| \leq \varepsilon/2$. Further, the upper semicontinuity of the spectrum implies that S can be chosen so that

$$\mathrm{dist}(\mathrm{\alpha_i}, \sigma(\mathrm{A})) \leq \varepsilon/2$$

for all $i \in \{1, \ldots, n\}$. By perturbing the elements $\alpha_1, \ldots, \alpha_n$ to be contained in $\sigma(A) \subseteq (0, \infty)$, we arrive at an operator $S' = \alpha'_1 P'_1 + \cdots + \alpha'_m P'_m$ where each α'_i is positive, $\sigma(S') = \{\alpha'_1, \ldots, \alpha'_m\}$, P'_i is the spectral projection of S' corresponding to $\{\alpha'_i\}$, and

$$\|S' - A\| \leq \|S' - S\| + \|S - A\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, the proof is complete.

■

As we shall see in Section 4, Corollary 3.3.5 together with the embedding results of Section 3.2 may be used to show that in a unital, simple, purely infinite C^*-algebra, any positive element with connected spectrum containing 0 is a norm limit of nilpotents. This result can then be extended to normal elements in such a C^*-algebra, thereby demonstrating an analogue of Theorem 2.2.1.

4 Normal Limits of Nilpotents in Purely Infinite C^*-Algebras

In Section 2, we presented a characterization of the closure of nilpotent operators in the Calkin algebra, $\mathcal{Q}(\mathcal{H})$ (in the case of a separable, infinite-dimensional Hilbert space, $\mathcal{H}$), which was seen in Corollary 3.2.12 to be an example of a unital, simple, purely infinite C^*-algebra. One may now ask whether other unital, simple, purely infinite C^*-algebras enjoy similar characterizations. As it turns out, simple necessary and sufficient conditions for a normal operator to lie within the closure of nilpotents in such an algebra do indeed exist. This characterization is due to P. Skoufranis [22] and is presented in Theorem 4.1.6, followed by an analysis of important consequences.

We begin by proving the following lemma which plays a key role in the proofs of Proposition 4.1.2 and Theorem 4.1.6, the main results of this section.

Lemma 4.1.1. *For each $n \in \mathbb{N}$, there exists a positive matrix $A_n \in \mathbb{M}_n(\mathbb{C})$ of unit norm such that $lim_{n\to\infty} \mathrm{dist}(A_n, \mathrm{Nil}(\mathbb{M}_n(\mathbb{C}))) = 0$.*

Proof. Recall from Lemma 2.2.2 that for each $m \geq 2$ there exists a norm 1, nilpotent matrix $M \in \mathbb{M}_{(2m+1)m+1}(\mathbb{C})$ and a normal matrix $N \in \mathbb{M}_{(2m+1)m+1}(\mathbb{C})$ such that $\|M - N\| \leq \pi/m + 1/m$ and

$$\sigma(N) = \left\{ \frac{k}{m} e^{\frac{i\pi}{m} j} : j \in \{1, \dots, 2m\},\ k \in \{0, \dots, m\} \right\}.$$

Let f be an analytic bijection from $\mathbb{D}$ onto

$$[0,1]_{1/2m} := \{z \in \mathbb{C} : \mathrm{dist}(z, [0,1]) < 1/2m\}$$

such that $f(0) = 0$. By approximating f uniformly by polynomials, we may obtain a polynomial, p, such that $p(0) = 0$ and $p(\mathbb{D}) \subseteq [0,1]_{1/m}$. Suppose that $p(z) = a_1 z + \cdots + a_t z^t$ for some $t \in \mathbb{N}$ and scalars $a_1, \dots, a_t$. Then $p(N)$ is normal, $p(M)$ is nilpotent, and $\sigma(p(N)) \subseteq [0,1]_{1/m}$. Let U_m be a unitary matrix in $\mathbb{M}_{(2m+1)m+1}(\mathbb{C})$ that diagonalizes $p(N)$, so that

$$U_m p(N) U_m^* = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_{(2m+1)m+1} \end{bmatrix} =: D.$$

Define the matrix $R_m \in \mathbb{M}_{(2m+1)m+1}(\mathbb{C})$ by

$$R_m := \begin{bmatrix} \mathrm{Re}(d_1) & & \\ & \ddots & \\ & & \mathrm{Re}(d_{(2m+1)m+1}) \end{bmatrix},$$

and note that for each $i \in \{1, \dots, (2m+1)m+1\}$, we have $|d_i - \mathrm{Re}(d_i)| < 1/m$, by construction. Notice also that by applying small perturbations to $\sigma(R_m)$, we may assume that $U_m^* R_m U_m$ is a positive, norm 1 element of $\mathbb{M}_{(2m+1)m+1}(\mathbb{C})$, and

$$\|p(N) - U_m^* R_m U_m\| = \|U_m p(N) U_m^* - R_m\| = \|D - R_m\| < \frac{1}{m}.$$

Since $\|N\|, \|M\| \leq 1$, it follows that $\|N^\ell - M^\ell\| \leq \ell \|N - M\|$ for all $\ell \in \mathbb{N}$. Thus,

$$\begin{aligned} \|p(N) - p(M)\| &\leq |a_1|\|N - M\| + |a_2|\|N^2 - M^2\| + \cdots |a_t|\|N^t - M^t\| \\ &\leq (|a_1| + 2|a_2| + \cdots + t|a_t|)\|N - M)\| \\ &= c\|N - M\| \end{aligned}$$

where $c := |a_1| + 2|a_2| + \cdots + t|a_t|$. Combining the above observations, it is evident that

$$\begin{aligned}\operatorname{dist}(U_m^* R_m U_m, \operatorname{Nil}(\mathbb{M}_{(2m+1)m+1}(\mathbb{C}))) &\leq \|U_m^* R_m U_m - p(M)\| \\ &\leq \|U_m^* R_m U_m - p(N)\| + \|p(N) - p(M)\| \\ &< \frac{1}{m} + c\left(\frac{\pi}{m} + \frac{1}{m}\right),\end{aligned}$$

and hence this distance tends to 0 as m becomes large. With this in mind, we define the sequence $\{A_n\}_{n\geq 1}$ as follows: For each $n \in \mathbb{N}$, let m_n denote the largest integer such that $2(m_n+1)m_n+1 \leq n$. By setting $\alpha := n - [(2m_n+1)m_n+1]$, we may define

$$A_n := U_{m_n}^* R_{m_n} U_{m_n} \oplus 0_{\alpha\times\alpha} \in \mathbb{M}_n(\mathbb{C})$$

and arrive at a sequence with the desired properties.

■

The following proposition due to Skoufranis [22] completely characterizes when a positive element in a unital, simple, purely infinite C^*-algebra belongs to the closure of nilpotents. Although this result is not necessary to prove Theorem 4.1.6, it does help to expose many of the technicalities that arise in its proof.

Proposition 4.1.2. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, and let $A \in \mathfrak{A}_+$. Then the following are equivalent:*

(1) $A \in \overline{\operatorname{Nil}(\mathfrak{A})}$.

(2) $A \in \overline{\operatorname{QNil}(\mathfrak{A})}$.

(3) $\sigma(A)$ *is connected and contains 0.*

Proof. It is obvious that (1) implies (2), and that (2) implies (3) follows from Newburgh's version of the upper semicontinuity of the spectrum. Let us suppose that (3) holds, so $\sigma(A)$ is connected and contains 0. Let $\varepsilon > 0$. We note that since $\mathfrak{A}$ is unital, simple, and purely infinite, $\mathfrak{A}$ has real rank zero, and so appealing to Corollary 3.3.5 allows us to find scalars $0 = a_n < a_{n-1} < \cdots < a_1 = \|A\|$ and non-zero, pairwise orthogonal projections $P_1^{(1)}, \ldots, P_n^{(1)}$, such that $\|A - A_1\| \leq \varepsilon$, where

$$A_1 := \sum_{i=1}^n a_i P_i^{(1)}.$$

Notice that $\sigma(A_1) = \{a_1, \ldots, a_n\}$, so by the assumption that $\sigma(A)$ is connected, together with Lemma 2.2.4 and upper semicontinuity of the spectrum, we may assume that

$$\max_{1\leq k\leq n-1} |a_{k+1} - a_k| < \varepsilon. \tag{$*$}$$

Turning to Lemma 4.1.1, we can find $\ell \in \mathbb{N}$, a positive matrix $T_1 \in \mathbb{M}_\ell(\mathbb{C})$ with $\|T_1\| = \|A\|$, and a nilpotent matrix $M_1 \in \mathbb{M}_\ell(\mathbb{C})$ such that $\|T_1 - M_1\| < \varepsilon$. Further, a small perturbation to the eigenvalues of T_1 allows us to assume that a_1 has multiplicity 1. For each $k \in \{2, \ldots, n\}$, let $\left\{\lambda_{1,k}, \ldots, \lambda_{m_k^{(1)},k}\right\}$ denote the (possibly empty) portion of $\sigma(T_1)$ that is contained in the interval $[a_k, a_{k-1})$, counting multiplicity. For each $k \in \{2, \ldots, n\}$, Theorem 3.2.8 guarantees the existence of pairwise orthogonal projections

$$Q_{1,k}^{(1)}, \ldots, Q_{m_k^{(1)},k}^{(1)}$$

whose sum is a proper subprojection of $P_k^{(1)}$ and such that $Q_{j,k}^{(1)}$ is Murray–von Neumann equivalent to $P_1^{(1)}$ for every $j \in \{1, \dots, m_k^{(1)}\}$. For each $k \in \{2, \dots, n\}$, define

$$P_k^{(2)} := P_k^{(1)} - \sum_{j=1}^{m_k^{(1)}} Q_{j,k}^{(1)},$$

and note that $P_k^{(2)}$ is a non-trivial projection for each choice of k. Further, define

$$A_1' := a_1 P_1^{(1)} + \sum_{k=1}^{n} \sum_{j=1}^{m_k^{(1)}} a_k Q_{j,k}^{(1)} \quad \text{and} \quad A_2 := \sum_{k=2}^{n} a_k P_k^{(2)}.$$

One can easily verify that

- $A_1', A_2 \in \mathfrak{A}_{sa}$ and $A_1 = A_1' + A_2$,
- $\sigma(A_2) = \{a_2, \dots, a_n\}$, and
- if we define $P^{(2)} := \sum_{k=2}^{n} P_k^{(2)}$, then $P^{(2)}$ is a non-trivial projection such that

$$A_1' = (I - P^{(2)}) A_1 (I - P^{(2)}) \ , \ \text{and} \ \ A_2 = P^{(2)} A_1 P^{(2)}.$$

Our goal is now to approximate A_1' within 2ε of a nilpotent inside the corner $(I - P^{(2)})\mathfrak{A}(I - P^{(2)})$. To do this, define

$$A_1'' := a_1 P_1^{(1)} + \sum_{k=1}^{n} \sum_{j=1}^{m_k^{(1)}} \lambda_{j,k} Q_{j,k}^{(1)} \in (I - P^{(2)})\mathfrak{A}(I - P^{(2)}),$$

and note that $(*)$ can be used to infer that $\|A_1'' - A_1'\| \le \varepsilon$. By construction, the collection

$$\left\{P_1^{(1)}\right\} \cup \left\{Q_{1,k}^{(1)}, \dots, Q_{m_k^{(1)},k}^{(1)} : k = 2, \dots, n\right\}$$

defines a set of pairwise orthogonal equivalent projections in $(I - P^{(2)})\mathfrak{A}(I - P^{(2)})$, and hence we may use the partial isometries implementing these equivalences to build a matrix algebra whose orthogonal minimal projections are precisely the projections described above. Moreover, A_1'' is a diagonal operator in this algebra with spectrum identical to that of T_1. Hence we may approximate A_1'' with the (still nilpotent) analogue of M_1 in $(I - P^{(2)})\mathfrak{A}(I - P^{(2)})$. Since $\|A_1'' - A_1'\| \le \varepsilon$, it follows that A_1' can be approximated within 2ε of a nilpotent in $(I - P^{(2)})\mathfrak{A}(I - P^{(2)})$, as advertised.

As A_2 is of the same form as A_1 but with the largest eigenvalue removed, we simply repeat the above analysis in the unital, simple, purely infinite corner $P^{(2)}\mathfrak{A}P^{(2)}$ to obtain a similar decomposition for A_2. By invoking this technique a finite number of times, we may express A_1 as a finite direct sum of operators, each within 2ε of a nilpotent. The direct sum of these nilpotents defines a nilpotent operator $M \in \mathfrak{A}$, and since $\|A_1 - A\| \le \varepsilon$, it follows that $\|A - M\| \le 3\varepsilon$, thereby completing the proof.

■

In order to extend this characterization to all normal elements of a unital, simple, purely infinite C^*-algebra, we will require a well-known result due to Lin [13] that provides a simple characterization of when a given normal operator can be approximated by normals with finite spectra. We use the notation $\mathrm{GL}(\mathfrak{A})_0$ to denote the connected component of $\mathrm{GL}(\mathfrak{A})$ that contains the identity.

Theorem 4.1.3 (Lin)**.** *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, and let $N \in \mathrm{Nor}(\mathfrak{A})$. Then N can be approximated by normals elements with finite spectra if and only if $N - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ for all $\lambda \in \mathbb{C} \setminus \sigma(N)$.*

Lemma 4.1.4. *Let $\mathfrak{A}$ be a unital C^*-algebra and $T \in \overline{\mathrm{QNil}(\mathfrak{A})}$. Then $T - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ for all $\lambda \in \mathbb{C} \setminus \sigma(T)$.*

Proof. Let $M \in \mathrm{QNil}(\mathfrak{A})$. Then $\sigma(tM) = t \cdot \sigma(M) = \{0\}$ for all $t \in \mathbb{C}$, and hence $tM - \lambda I \in \mathrm{GL}(\mathfrak{A})$ for all $t, \lambda \in \mathbb{C}$ with $\lambda \neq 0$. It then follows that $M - \lambda I$ can be connected to I by a continuous path in $\mathrm{GL}(\mathfrak{A})$ whenever $\lambda \neq 0$, and hence $M - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Now suppose that $T \in \overline{\mathrm{QNil}(\mathfrak{A})}$ and let $\lambda \in \mathbb{C} \setminus \sigma(T)$. Let $\{M_n\}_{n \geq 1}$ be a sequence in $\mathrm{QNil}(\mathfrak{A})$ converging to T, and note that clearly

$$T - \lambda I = \lim_{n \to \infty} (M_n - \lambda I).$$

The semicontinuity of the spectrum implies that $0 \in \sigma(T)$, and hence $\lambda \neq 0$. By the above analysis, $M_n - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ for all $n \in \mathbb{N}$, and since $\mathrm{GL}(\mathfrak{A})_0$ is open, we conclude that $T - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ as well.

■

As we will soon see, the proof of Theorem 4.1.6 relies on the connectedness of a certain graph that arises from the spectrum of a given normal operator, which will follow from simple graph theoretic arguments. Since this result sees repeated application in the sections to come, we present it here for completeness.

Lemma 4.1.5. *If G is a connected graph with $n \geq 2$ vertices, then the set of vertices of G whose removal keeps G connected has cardinality at least 2.*

Proof. Let us call a vertex *good* for G if it's removal does not disconnect G, and *bad* otherwise. Our goal is to show that G has at least 2 good vertices, which we shall achieve by an inductive argument on the number of vertices, n. The case for $n = 2$ is obvious and the case $n = 3$ is shown below. Here we see the only two connected graphs on $n = 3$ vertices, where the black vertices indicate those that are good for the given graph.

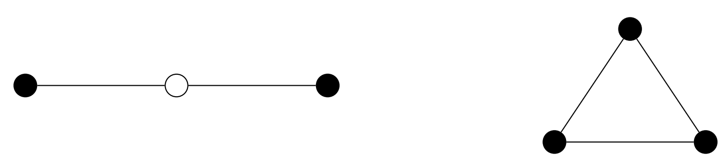

Figure 4: $n = 3$

Now suppose that every connected graph with at least two and at most n vertices possesses at least two good vertices. Let G be a connected graph with $n+1$ vertices. If every vertex is good for G, then we are done. Otherwise, let v denote a bad vertex for G, and for simplicity, assume that the removal of vertex v disconnects G into two connected components, G_1 and G_2. We will show that both of these subgraphs contain vertices that are good for G.

If G_1 has only one vertex, then clearly this vertex is good for G. If G_1 has at least two vertices, then the inductive assumption implies that there exist vertices a, b in G_1 that are good for G_1. To arrive at a contradiction, suppose that both a and b are bad for G. Since neither of these vertices can connect to G_2 and both are good for G_1, it must be the case that a and b were connected to v. Notice that if we had removed a from G, then b would connect to every vertex in G_2 via v, and also to everything that remains in G_1, as b was good for G_1. Hence, removing a from G yields a connected graph. This is a contradiction, and hence either a or b must be a good vertex for G. A symmetric argument will also allow us to obtain a good vertex from G_2, and hence G contains at least 2 good vertices. The result now follows by induction.

■

Theorem 4.1.6 (Skoufranis)**.** *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let N be a normal element of $\mathfrak{A}$. The following are equivalent:*

(1) $N \in \overline{\mathrm{Nil}(\mathfrak{A})}$.

(2) $N \in \overline{\mathrm{QNil}(\mathfrak{A})}$.

(3) $\sigma(N)$ *is connected,* $0 \in \sigma(N)$*, and* $N - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ *for all* $\lambda \in \mathbb{C} \setminus \sigma(N)$.

Proof. It is clear that (1) implies (2), and (2) implies (3) follows from the semicontinuity of the spectrum together with Lemma 4.1.4. Suppose that the assumptions in (3) hold. Given $\varepsilon > 0$, Theorem 4.1.3 produces a normal operator N_ε with finite spectrum such that $\|N - N_\varepsilon\| \leq \varepsilon$.

For each pair $(n, m) \in \mathbb{Z}^2$, let

$$B_{n,m} := \left(\varepsilon n - \frac{\varepsilon}{2}, \varepsilon n + \frac{\varepsilon}{2}\right] + i\left(\varepsilon m - \frac{\varepsilon}{2}, \varepsilon m + \frac{\varepsilon}{2}\right].$$

We shall call a box $B_{n,m}$ *relevant* if $\sigma(N_\varepsilon) \cap B_{n,m} \neq \emptyset$. Define $f : \mathbb{C} \to \mathbb{C}$ to be the function that sends an element $\lambda \in \mathbb{C}$ to the center of the box that contains λ. Since the spectrum of N_ε is discrete, we have that f is continuous on $\sigma(N_\varepsilon)$ and hence $f(N_\varepsilon)$ defines an element of $\mathrm{Nor}(\mathfrak{A})$ whose spectrum is precisely the centers of the relevant boxes. Moreover,

$$\|N - f(N_\varepsilon)\| \leq \|N - N_\varepsilon\| + \|N_\varepsilon - f(N_\varepsilon)\| \leq 2\varepsilon,$$

so by replacing N_ε by $f(N_\varepsilon)$, we may assume that $\sigma(N_\varepsilon)$ is exactly the centers of the relevant boxes and $\|N - N_\varepsilon\| \leq 2\varepsilon$. Notice that since $\sigma(N)$ was assumed to be connected, the upper semicontinuity of the spectrum coupled with Lemma 2.2.4 allows us to assume that the union of relevant boxes is connected and that $B_{0,0}$ is relevant. In the language of Lemma 4.1.5, we call a relevant box *bad* if it is $B_{0,0}$ or its removal disconnects the union of relevant boxes, and *good* otherwise.

If $B_{0,0}$ is the only relevant box, then $N_\varepsilon = 0$ which is clearly nilpotent, and the proof is complete. Otherwise let B_{n_0,m_0} be a good, relevant box (which exists by Lemma 4.1.5). Since the union of relevant boxes is connected, we can find a continuous function, γ, from $[0, 1]$ into the union of relevant boxes such that $\gamma(0) = 0$ and $\gamma(1) = \varepsilon n_0 + i\varepsilon m_0$. The Stone-Weierstrass theorem guarantees that γ can be approximated uniformly on $[0, 1]$ by a polynomial that vanishes at 0. Let p be such a polynomial and assume that

$$\sup_{t \in [0,1]} |p(t) - \gamma(t)| \leq \varepsilon.$$

Given $\delta > 0$, an application of Lemma 4.1.1 yields a positive intger ℓ, a positive matrix N_ℓ of unit norm in $\mathbb{M}_\ell(\mathbb{C})$, and a nilpotent matrix M_ℓ in $\mathbb{M}_\ell(\mathbb{C})$ such that $\|N_\ell - M_\ell\| \leq \delta$. By replacing N_ℓ and M_ℓ by their respective images under p and by selecting an appropriately small value of δ, we may assume that $\sigma(N_\ell)$ is contained in an ε-neighbourhood of the union of relevant boxes, and $\|N_\ell - M_\ell\| \leq \varepsilon$. We now perform small perturbations to the eigenvalues of N_ℓ so that

- $\sigma(N_\ell) \subseteq \sigma(N_\varepsilon)$, and
- the algebraic multiplicity of $\varepsilon n_0 + i\varepsilon m_0$ is exactly 1.

Since each of these perturbations can be carried out at a cost of at most 2ε, it is reasonable to assume that N_ℓ satisfies the above properties and

$$\|N_\ell - M_\ell\| \leq \varepsilon + 2\varepsilon + 2\varepsilon = 5\varepsilon.$$

Given a pair $(n, m) \in \mathbb{Z}^2$, let $P_{n,m}$ denote the spectral projection of N_ℓ corresponding to the box $B_{n,m}$, and let $\alpha_{n,m}$ denote the algebraic multiplicity of $\varepsilon n + i\varepsilon m$ for the operator N_ℓ. Note that whenever $(n, m) \in \mathbb{Z}^2$ is such that $B_{n,m}$ is relevant and not B_{n_0,m_0}, Theorem 3.2.8 allows us to find $\alpha_{n,m}$ pairwise orthogonal subprojections of $P_{n,m}$ whose sum is a proper subprojection of $P_{n,m}$ and such that each is Murray–von Neumann equivalent to P_{n_0,m_0}. We now mimic the proof of Proposition 4.1.2 to obtain a projection $P_1 \in \mathfrak{A}$ such that

- $P_1 N_\varepsilon P_1$ can be approximated within 5ε by a nilpotent in $P_1 \mathfrak{A} P_1$, and
- $(I - P_1)N_\varepsilon(I - P_1)$ has spectrum equal to $\sigma(N_\varepsilon) \setminus \{\varepsilon n_0 + i\varepsilon m_0\}$.

By construction, the number of relevant boxes for $(I - P_1)N_\varepsilon(I - P_1)$ is exactly one less than for N_ε, and the union of relevant boxes is still connected. We now repeat the above analysis with N_ε replaced by $(I - P_1)N_\varepsilon(I - P_1)$. After a finite number of repetitions, we arrive at a nilpotent operator $M \in \mathfrak{A}$ such that $\|N_\varepsilon - M\| \leq 5\varepsilon$, and hence

$$\|N - M\| \leq \|N - N_\varepsilon\| + \|N_\varepsilon - M\| \leq 2\varepsilon + 5\varepsilon = 7\varepsilon,$$

completing the proof.

■

Before examining an important corollary of Theorem 4.1.6, we recall the following result due to Herrero, a proof of which can be found in [11]:

Theorem 4.1.7 (Herrero)**.** *Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space, and N be a normal operator in $\mathcal{B}(\mathcal{H})$. Then*

$$N \in \overline{\{M_1 + M_2 : M_1, M_2 \in \mathrm{Nil}(\mathcal{B}(\mathcal{H}))\}}.$$

As a consequence of Theorem 4.1.6 (or Proposition 4.1.2), one can show that an analogue of Herrero's result holds in the setting of a unital, simple, purely infinite C^*-algebra. However, as we will see in the following section, this result does *not* have a direct analogue in the case of a unital AF C^*-algebra.

Corollary 4.1.8. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra. Then*

$$\mathfrak{A}_{sa} \subseteq \overline{\{M_1 + M_2 : M_1, M_2 \in \mathrm{Nil}(\mathfrak{A})\}}$$

and

$$\mathfrak{A} = \overline{\{M_1 + M_2 + M_3 + M_4 : M_1, M_2, M_3, M_4 \in \mathrm{Nil}(\mathfrak{A})\}}.$$

Proof. For simplicity, define $\mathcal{N}_2(\mathfrak{A}) := \{M_1 + M_2 : M_1, M_2 \in \mathrm{Nil}(\mathfrak{A})\}$. Notice that since any element of $\mathfrak{A}$ can be decomposed into a sum of its real and imaginary parts, both of which are self-adjoint, the second claim will follow immediately once the first claim has been proven. To see that the first claim holds, we begin by demonstrating that the identity in any unital, simple, purely infinite C^*-algebra must belong to $\overline{\mathcal{N}_2(\mathfrak{A})}$. Let $A \in \mathfrak{A}_+$ be such that $\sigma(A) = [0,1]$. Since $\sigma(A)$ and $\sigma(I - A)$ are connected and contain 0, Proposition 4.1.2 ensures that both A and $I - A$ belong to $\overline{\mathrm{Nil}(\mathfrak{A})}$. Hence, it follows that

$$I = A + (I - A) \in \overline{\mathcal{N}_2(\mathfrak{A})}.$$

Now suppose that $A \in \mathfrak{A}_{sa}$ and let $\varepsilon > 0$. Since $\mathfrak{A}$ has real rank zero, we may apply Corollary 3.3.5 to obtain scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, and non-zero pairwise orthogonal projections $P_1, \ldots, P_n$ in $\mathfrak{A}$ such that

$$\left\| A - \sum_{k=1}^{n} \alpha_k P_k \right\| < \varepsilon. \tag{$*$}$$

However, each P_k is the identity in the corner $P_k \mathfrak{A} P_k$ (which is simple and purely infinite by results in Section 3), and hence for every $k \in \{1, \ldots, n\}$, we see that $P_k \in \overline{\mathcal{N}_2(P_k \mathfrak{A} P_k)}$. Finally, by considering the n-fold direct sum of elements from each set $\mathcal{N}_2(P_k \mathfrak{A} P_k)$ that approximate P_k, it is clear that $\sum_{k=1}^{n} \alpha_k P_k$ is contained in $\overline{\mathcal{N}_2(\mathfrak{A})}$, and we may conclude by $(*)$ that $A \in \overline{\mathcal{N}_2(\mathfrak{A})}$ as well. ■

Corollary 4.1.9. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, and let $N \in \mathrm{Nor}(\mathfrak{A})$ be such that $N - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ for all $\lambda \in \mathbb{C} \setminus \sigma(N)$. Then*

$$N \in \overline{\{M_1 + M_2 : M_1, M_2 \in \mathrm{Nil}(\mathfrak{A})\}}.$$

Proof. Since $N - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ for all $\lambda \in \mathbb{C} \setminus \sigma(N)$, an application of Theorem 4.1.3 produces a normal operator N' that approximates N and has finite spectrum. Let $\alpha_1, \ldots, \alpha_n$ be the distinct elements in $\sigma(N')$, and for each $i \in \{1, \ldots, n\}$, let P_i denote the spectral projection of N' corresponding to $\{\alpha_i\}$. Then

$$N' = \sum_{i=1}^{n} \alpha_i P_i,$$

and we may now apply the same arguments as in the proof of Corollary 4.1.8 to achieve the desired result.

■

5 C^*-Algebras with Tracial States

As we've seen in the preceding sections, there exist several theorems that allow us to characterize many, if not all elements that make up the closure of nilpotents in certain C^*-algebras. One would hope that these results extend to more general classes of C^*-algebras, but as we shall see in this section (which is based on recent work of P. Skoufranis [21]), some difficulties arise when the C^*-algebra possesses a tracial state. Section 5.1 will outline some of the obstructions that are faced in these settings and section 5.2 will demonstrate that strong positive results can still be obtained in the presence of a tracial state.

5.1 Obstructions by Tracial States

Definition 5.1.1. *Let $\mathfrak{A}$ be a C^*-algebra. A* ***tracial state*** *on $\mathfrak{A}$ is a positive linear functional τ such that $\|\tau\| = 1$ and $\tau(AB) = \tau(BA)$ for all $A, B, \in \mathfrak{A}$. A tracial state τ on $\mathfrak{A}$ is called* ***faithful*** *if $\tau(A) > 0$ whenever A is a non-zero element of $\mathfrak{A}_+$.*

Examples of C^*-algebras that possess faithful tracial states include finite-dimensional C^*-algebras (i.e., those isomorphic to finite direct sums of matrix algebras) and UHF C^*-algebras. A faithful tracial state is, in fact, a special case of what is called *a separating family of tracial states*, defined below.

Definition 5.1.2. *A C^*-algebra $\mathfrak{A}$ is said to possess a* ***separating family of tracial states*** *if for every $A \in \mathfrak{A}_+ \setminus \{0\}$, there exists a tracial state τ on $\mathfrak{A}$ such that $\tau(A) > 0$.*

In order to see why tracial states impose difficulty in approximation by nipotents, we turn to a result known as *Rota's theorem* (see [18]). The proof of this result as well as the lemma preceding it can be also found in [15].

Lemma 5.1.3. *Let X be an element of a unital C^*-algebra $\mathfrak{A}$ with $r(X) < 1$. Then there exists an element $Y \in \mathrm{GL}(\mathfrak{A})$ such that $\|YXY^{-1}\| < 1$.*

Proof. Note that since $r(X) < 1$, the series

$$\sum_{n=0}^{\infty} \|(X^n)^* X^n\| = \sum_{n=0}^{\infty} \|X^n\|^2$$

is convergent by the root test and the Beurling spectral radius formula. Thus, $Z := \sum_{n=0}^{\infty}(X^n)^* X^n$ defines an element of $\mathfrak{A}$. Moreover, since

$$Z - I = \sum_{n=1}^{\infty}(X^n)^* X^n \geq 0,$$

it follows that $Z \geq I$. Thus, $Y := Z^{1/2} \geq I$ (as $\sigma(Z) \subseteq [1, \infty)$ implies that $\sigma(Z^{1/2}) \subseteq [1, \infty)$, by the spectral mapping theorem), and hence $Y \in \mathrm{GL}(\mathfrak{A})$. Finally, the spectral mapping theorem may be used to show that $\sigma(I - Y^{-2}) \subseteq [0, 1)$, from which we conclude that

$$\begin{aligned}
\|YXY^{-1}\|^2 &= \|Y^{-1}X^*Y^2XY^{-1}\| \\
&= \|Y^{-1}(Z - I)Y^{-1}\| \\
&= \|I - Y^{-2}\| \\
&= r(I - Y^{-2}) < 1,
\end{aligned}$$

completing the proof. ∎

Theorem 5.1.4 (Rota's Theorem)**.** *If $\mathfrak{A}$ is a unital C^*-algebra and $X \in \mathfrak{A}$, then*

$$r(X) = \inf_{A \in \mathfrak{A}_{sa}} \|e^A X e^{-A}\| = \inf_{Y \in \mathrm{GL}(\mathfrak{A})} \|YXY^{-1}\|.$$

Proof. Choose $\delta > 0$ such that $r(X\delta^{-1}) < 1$, and apply the previous lemma to obtain an element $Y \in \mathrm{GL}(\mathfrak{A})$ such that $\|YXY^{-1}\| < \delta$. If $Y = U|Y|$ is the polar decomposition of Y, then $U = Y|Y|^{-1}$ is a unitary in $\mathfrak{A}$. Further, since $|Y|$ is invertible, we may choose $\varepsilon > 0$ such that $\sigma(|Y|) \subseteq [\varepsilon, \infty)$. It then follows that $A := \ln(|Y|)$ defines an element of $\mathfrak{A}_{sa}$, and since U is unitary, we see that

$$\|e^A X e^{-A}\| = \|Ue^A X e^{-A} U^*\| = \|YXY^{-1}\| < \delta.$$

When δ approaches $r(X)$, by considering infima in $\mathfrak{A}_{sa}$ and $\mathrm{GL}(\mathfrak{A})$, respectively, we obtain

$$\inf_{A \in \mathfrak{A}_{sa}} \|e^A X e^{-A}\|, \inf_{Y \in \mathrm{GL}(\mathfrak{A})} \|YXY^{-1}\| \leq r(X).$$

For the reverse inequalities, note that for every $A \in \mathfrak{A}$, we have $\sigma(X) = \sigma(e^A X e^{-A})$, by similarity. Hence,

$$r(X) = r(e^A X e^{-A}) \leq \|e^A X e^{-A}\|,$$

so the first equality holds by considering the infimum over $\mathfrak{A}_{sa}$. Moreover, given $Y \in \mathrm{GL}(\mathfrak{A})$, the same argument shows that

$$r(X) = r(YXY^{-1}) \leq \|YXY^{-1}\|.$$

The result now follows as above.

■

Corollary 5.1.5. *Let $\mathfrak{A}$ be a C^*-algebra with a tracial state, τ. If $M \in \overline{\mathrm{QNil}(\mathfrak{A})}$, then $\tau(M) = 0$.*

Proof. Let $\varepsilon > 0$. Since τ is continuous, we may assume that $M \in \mathrm{QNil}(\mathfrak{A})$. Further, since we can always extend τ to a tracial state on the unitization of $\mathfrak{A}$ by defining $\tilde{\tau}((A, \lambda)) := \lambda + \tau(A)$, we may also assume that $\mathfrak{A}$ is unital. Applying Rota's theorem, we can find an element $B \in \mathrm{GL}(\mathfrak{A})$ such that $\|BMB^{-1}\| < r(M) + \varepsilon = \varepsilon$, as $\sigma(M) = \{0\}$. But then

$$|\tau(M)| = |\tau(BMB^{-1})| \leq \|\tau\|\|BMB^{-1}\| < \varepsilon.$$

This proves that $\tau(M) = 0$, as desired.

■

Corollary 5.1.6. *Let $\mathfrak{A}$ be a C^*-algebra and let τ be a tracial state on $\mathfrak{A}$. Then*

$$\mathrm{dist}(A, \mathrm{QNil}(\mathfrak{A})) \geq |\tau(A)|$$

for every $A \in \mathfrak{A}$.

Proof. This is immediate from Corollary 5.1.5. Indeed, since $\tau(M) = 0$, we have

$$\mathrm{dist}(A, \mathrm{QNil}(\mathfrak{A})) = \inf_{M \in \mathrm{QNil}(\mathfrak{A})} \|A - M\| \geq \inf_{M \in \mathrm{QNil}(\mathfrak{A})} |\tau(A - M)| = |\tau(A)|.$$

■

As these results indicate, there are additional necessary conditions for an operator to be a norm limit of quasinilpotents whenever the algebra in question bears the burdern of a tracial state. The limitations these conditions impose can be seen in the results that follow.

Proposition 5.1.7. *Let $\mathfrak{A}$ be a C^*-algebra with a separating family of tracial states, and let $N \in \mathrm{Nor}(\mathfrak{A})$. Suppose that there exists a polynomial p such that*

1. *$p(0) = 0$,*
2. *$p(N) \neq 0$, and*
3. *$p(\sigma(N)) \subseteq [0, \infty)$.*

Then $N \notin \overline{\mathrm{QNil}(\mathfrak{A})}$. Consequently, $\mathfrak{A}_{sa} \cap \overline{\mathrm{QNil}(\mathfrak{A})} = \{0\}$.

Proof. Suppose to the contrary that there is an $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QNil}(\mathfrak{A})}$ and a polynomial p with the properties listed above. Since $p(0) = 0$, we have that $p(\mathrm{QNil}(\mathfrak{A})) \subseteq \mathrm{QNil}(\mathfrak{A})$, and hence $p(N) \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QNil}(\mathfrak{A})}$. Further, $p(N) > 0$, so by assumption we can find a tracial state τ on $\mathfrak{A}$ such that $\tau(p(N)) > 0$. But this clearly contradicts Corollary 5.1.5, and hence the result holds. For the final claim, note that if $N \in \mathfrak{A}_{sa} \backslash \{0\}$, then one can see that the polynomial $p(z) = z^2$ satisfies the three conditions above, and hence N cannot belong to $\overline{\mathrm{QNil}(\mathfrak{A})}$.

∎

Proposition 5.1.7 can be easily extended using Mergelyan's theorem. The statement of this result is given below and a proof can be found in [19, Theorem 20.5].

Theorem 5.1.8 (Mergelyan's Theorem)**.** *Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected. If f is a function that is continuous on K and analytic on $\mathrm{int}(K)$, then f can be approximated uniformly by polynomials on K.*

Corollary 5.1.9. *Let $\mathfrak{A}$ be a C^*-algebra and $N \in \mathrm{Nor}(\mathfrak{A}) \setminus \{0\}$ be such that $\mathrm{int}(\sigma(N)) = \emptyset$ and $\mathbb{C} \setminus \sigma(N)$ is connected. Then*

1. *$N \notin \overline{\mathrm{QNil}(\mathfrak{A})}$ whenever $\mathfrak{A}_+ \cap \overline{\mathrm{QNil}(\mathfrak{A})} = \{0\}$, and*
2. *$N \notin \overline{\mathrm{Nil}(\mathfrak{A})}$ whenever $\mathfrak{A}_+ \cap \overline{\mathrm{Nil}(\mathfrak{A})} = \{0\}$.*

As a result, if $\mathfrak{A}$ is a C^*-algebra with a separating family of tracial states, then $N \notin \overline{\mathrm{QNil}(\mathfrak{A})}$.

Proof. We shall only prove (1) as the proof of (2) is identical. Suppose that $\mathfrak{A}_+ \cap \overline{\mathrm{QNil}(\mathfrak{A})} = \{0\}$ and $N \in \overline{\mathrm{QNil}(\mathfrak{A})}$. If we define $f(z) = |z|$, then $f \in \mathcal{C}(\sigma(N))$, and so Mergelyan's theorem implies that f is a uniform limit of polynomials on $\sigma(N)$. Moreover, the semicontinuity of the spectrum guarantees that $0 \in \sigma(N)$, and since $f(0) = 0$, we may assume that the polynomials described above all vanish at 0. Hence $f(N) \in \overline{\mathrm{QNil}(\mathfrak{A})}$, but the spectral mapping theorem also shows that $f(N) \in \mathfrak{A}_+$. By assumption $f(N) = 0$, and hence $N = 0$. Finally, if $\mathfrak{A}$ has a separating family of tracial states, then $\mathfrak{A}_{sa} \cap \overline{\mathrm{QNil}(\mathfrak{A})} = \{0\}$ by Proposition 5.1.7. Since

$$\mathfrak{A}_+ \cap \overline{\mathrm{QNil}(\mathfrak{A})} \subseteq \mathfrak{A}_{sa} \cap \overline{\mathrm{QNil}(\mathfrak{A})} = \{0\},$$

the problem reduces to the first case above.

∎

Since every finite-dimensional C^*-algebras sees the existence of a faithful tracial state, it becomes interesting to ask whether the obstructions caused by these tracial states extend to direct limits of these algebras. As it turns out, some troubles do arise. The remainder of this section will be devoted to exploring the difficulties tracial states impose for approximation by nilpotent operators in unital AF C^*-algebras. We begin with the following proposition which provides a useful characterization of the closure of nilpotents in this setting.

Proposition 5.1.10. *Let $\mathfrak{A} = \overline{\bigcup_{k\geq 1} \mathfrak{A}_k}$ be an AF C^*-algebra, where each $\mathfrak{A}_k$ is a finite-dimensional C^*-algebra. If $T \in \mathfrak{A}$, then TFAE:*

(1) $T \in \overline{\mathrm{QNil}(\mathfrak{A})}$.

(2) $T \in \overline{\mathrm{Nil}(\mathfrak{A})}$.

(3) $T \in \overline{\bigcup_{k\geq 1} \mathrm{Nil}(\mathfrak{A}_k)}$.

Proof. Since (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious, we shall only discuss the implication (1) $\Rightarrow$ (3). Assume that $T \in \overline{\mathrm{QNil}(\mathfrak{A})}$. Let $\varepsilon > 0$ and choose $M \in \mathrm{QNil}(\mathfrak{A})$ such that $\|T - M\| < \varepsilon$. Since $M \in \overline{\cup_{k\geq 1}\mathfrak{A}_k}$, we may choose $k \in \mathbb{N}$ and $M_0 \in \mathfrak{A}_k$ with $\|M - M_0\| < \varepsilon$. Further, the upper semicontinuity of the spectrum allows us to assume that

$$\sigma(M_0) \subseteq \{z \in \mathbb{C} : \mathrm{dist}(z, \sigma(M)) < \varepsilon\} = B_\varepsilon(0),$$

as $M \in \mathrm{QNil}(\mathfrak{A})$. Since $\mathfrak{A}_k$ is finite-dimensional, there is a unitary U such that UM_0U^* is upper-triangular. Letting D denote the diagonal portion of UM_0U^*, and M_0' denote the part of UM_0U^* strictly above the diagonal, it is clear that M_0' is nilpotent and $\sigma(D) = \sigma(M_0) \subseteq B_\varepsilon(0)$, hence $\|UM_0U^* - M_0'\| = \|D\| < \varepsilon$. Thus, we see that

$$\|T - U^*M_0'U\| \leq \|T - M\| + \|M - M_0\| + \|M_0 - U^*M_0'U\| < 3\varepsilon.$$

Since $U^*M_0'U \in \mathrm{Nil}(\mathfrak{A}_k)$, the result holds. ∎

This result, together with its predecessors, can now be used to show that Theorem 4.1.7 and Corollary 4.1.8 do not see direct analogues in the case of a unital AF C^*-algebra.

Lemma 5.1.11. *Let $\mathfrak{A} = \overline{\bigcup_{k\geq 1} \mathrm{Nil}(\mathfrak{A}_k)}$ be a unital AF C^*-algebra where each $\mathfrak{A}_k$ is finite-dimensional. For an element $T \in \mathfrak{A}$, each of the following sets contains at most 1 element:*

(1) $\{\lambda \in \mathbb{C} : \lambda I + T \in \overline{\mathrm{Nil}(\mathfrak{A})}\}$,

(2) $\{\lambda \in \mathbb{C} : \lambda I + T \in \overline{\mathrm{span}(\mathrm{Nil}(\mathfrak{A}))}\}$.

Proof. We present only the proof for (1), as (2) follows similarly. Suppose that $\lambda_0 \in \mathbb{C}$ is such that $\lambda_0 I + T \in \overline{\mathrm{Nil}(\mathfrak{A})}$, and define $R := \lambda_0 I + T$. In order to show that $\lambda I + T \notin \overline{\mathrm{Nil}(\mathfrak{A})}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$, it suffices to show that $\mu I + R \notin \overline{\mathrm{Nil}(\mathfrak{A})}$ for all $\mu \neq 0$. Suppose that $\mu \in \mathbb{C}$ is such that $\mu I + R \in \overline{\mathrm{Nil}(\mathfrak{A})}$. For each $k \in \mathbb{N}$, choose $R_k \in \mathfrak{A}_k$ such that $R = \lim_k R_k$, and note that of course this implies that $\mu I + R = \lim_k(\mu I + R_k)$. Since R and $\mu I + R$ belong to $\overline{\mathrm{Nil}(\mathfrak{A})}$, by applying Proposition 5.1.10 we can find sequences $\{M_k\}_{k\geq 1}$ and $\{M_k'\}_{k\geq 1}$ such that

- $R = \lim_k M_k$,
- $\mu I + R = \lim_k M_k'$, and
- $M_k, M_k' \in \mathrm{Nil}(\mathfrak{A}_k)$ for each k.

Hence $\lim_k \|R_k - M_k\| = \lim_k \|(\mu I + R_k) - M_k'\| = 0$. For each $k \geq 1$, let τ_k be a normalized, faithful tracial state on $\mathfrak{A}_k$. Then

$$|\tau_k(R_k)| = |\tau_k(R_k - M_k)| \leq \|R_k - M_k\| \to 0,$$

and similarly,

$$|\tau_k(\mu I + R_k)| = |\tau_k((\mu I + R_k) - M_k')| \leq \|(\mu I + R_k) - M_k'\| \to 0.$$

By these observations together with the properties of the chosen tracial states, we may now conclude that

$$\mu = \lim_k \tau_k(\mu I + R_k) = 0,$$

and the proof is complete.

■

Corollary 5.1.12. *Let $\mathfrak{A}$ be a unital AF C^*-algebra. Then $I \notin \overline{\mathrm{span}(\mathrm{Nil}(\mathfrak{A}))}$.*

Proof. Since $0 = 0 \cdot I + 0 \in \overline{\mathrm{span}(\mathrm{Nil}(\mathfrak{A}))}$, the fact that the second set in Lemma 5.1.11 has cardinality at most 1 implies that $I = 1 \cdot I + 0$ cannot belong to $\overline{\mathrm{span}(\mathrm{Nil}(\mathfrak{A}))}$.

■

5.2 Normal Limits of Nilpotents in UHF C^*-Algebras

Although UHF C^*-algebras do possess faithful tracial states and hence suffer some of the drawbacks presented in Section 5.1, we note that positive results are still within reach. In particular, the following theorem due to Marcoux (which was communicated to Skoufranis and can be found in [21]) demonstrates that in any infinite-dimensional UHF C^*-algebra, there is a normal operator in the closure of nilpotents with spectrum given by the closed unit disk. Classical results of complex analysis and spectral theory (similar to those mentioned in Section 2) can then be used to unearth a plethora of normal operators that belong to the closure of nilpotents in these algebras.

Theorem 5.2.1 (Marcoux)**.** *Let $\mathfrak{A}$ be an infinite-dimensional UHF C^*-algebra. Then there exists a normal element N in $\overline{\mathrm{Nil}(\mathfrak{A})}$ such that $\sigma(N) = \overline{\mathbb{D}}$.*

Proof. Assume that $\mathfrak{A} = \overline{\bigcup_{k\geq 1} \mathbb{M}_{\ell_k}(\mathbb{C})}$, where each $\mathbb{M}_{\ell_k}$ embeds unitally into $\mathbb{M}_{\ell_{k+1}}$. Since $\mathfrak{A}$ is not finite-dimensional, we may assume that $\ell_1 \geq 11$ and ℓ_{k+1}/ℓ_k is composite for all $k \geq 1$. Our goal is to define a Cauchy sequence of normal matrices $(N_k)_{k\geq 1}$ with $N_k \in \mathbb{M}_{\ell_k}$ for each $k \geq 1$, and whose limit is the normal element N as described in the statement of the theorem. We will define this sequence inductively by finding, for each $k \geq 1$, elements $m_k, n_k \in \mathbb{N}$, and $q_k \in \mathbb{N} \cup \{0\}$ such that

- $m_1, n_1 \geq 2$,
- $m_{k+1} \geq 2m_k$, $n_{k+1} \geq 2n_k$ for all $k \geq 1$, and
- $\ell_k = (2m_k + 1)n_k + 1 + q_k$ for all $k \geq 1$

We first define m_1, n_1, q_1, and N_1. Let $m_1 = 2$ and choose $n_1 \geq 2$ and $q_1 \in \{0, 1, 2, 3, 4\}$ so that $\ell_1 = (2m_1 + 1)n_1 + 1 + q_1$. Now let N_1 be the direct sum of the $q_1 \times q_1$ zero matrix with the normal matrix obtained from Lemma 2.2.2 by setting $m = m_1$ and $n = n_1$.

Now suppose we have constructed N_k for some fixed $k \in \mathbb{N}$, as well as m_k, n_k, and q_k so that the above three properties are satisfied. Since ℓ_{k+1}/ℓ_k was assumed to be composite, we can find $p, z \geq 2$ so that $\ell_{k+1}/\ell_k = pz$. Hence, every eigenvalue of N_k has pz-times its multiplicity when we view N_k as an element of $\mathbb{M}_{\ell_{k+1}}(\mathbb{C})$. We now define $m_{k+1} := zm_k \geq 2m_k$ and $n_{k+1} := pn_k \geq 2n_k$. One can easily verify from the fact that $\ell_k = (2m_k + 1)n_k + 1 + q_k$, that

$$(2m_{k+1} + 1)n_{k+1} + 1 + ((z-1)pn_k + pz + pzq_k - 1) = (pz)\ell_k = \ell_{k+1},$$

and so by defining $q_{k+1} := (z-1)pn_k + pz + pzq_k - 1$, we see that $\ell_{k+1} = (2m_{k+1}+1)n_{k+1} + 1 + q_{k+1}$, as desired. Let N'_{k+1} be the direct sum of the $q_{k+1} \times q_{k+1}$ zero matrix with the normal matrix obtained from Lemma 2.2.2 by setting $m = m_{k+1}$ and $n = n_{k+1}$.

It is evident from the description of the spectrum from Lemma 2.2.2 that we can pair the eigenvalues of N_k (when viewed as an element of $\mathbb{M}_{\ell_{k+1}}(\mathbb{C})$) with those of N'_{k+1} so that for every pair (λ, μ), we have

$$|\lambda - \mu| < \frac{\pi}{n_k} + \frac{1}{m_k}.$$

Thus, by considering the diagonalizations of N_k and N'_{k+1}, it follows from the above that there exists a unitary $U \in \mathbb{M}_{\ell_{k+1}}(\mathbb{C})$ such that $N_{k+1} := UN'_{k+1}U^*$ satisfies

$$\|N_{k+1} - N_k\| < \frac{\pi}{n_k} + \frac{1}{m_k}.$$

Moreover, since $m_{k+1} \geq 2m_k$ and $n_{k+1} \geq 2n_k$ for all $k \geq 1$, we see that for $s \geq k$,

$$\begin{aligned}
\|N_{s+1} - N_k\| &\leq \|N_{s+1} - N_s\| + \|N_s - N_{s-1}\| + \cdots + \|N_{k+1} - N_k\| \\
&< \left(\frac{\pi}{n_s} + \frac{1}{m_s}\right) + \left(\frac{\pi}{n_{s-1}} + \frac{1}{m_{s-1}}\right) + \cdots + \left(\frac{\pi}{n_k} + \frac{1}{m_k}\right) \\
&\leq \left(\frac{\pi}{2^{s-k}n_k} + \frac{1}{2^{s-k}m_k}\right) + \left(\frac{\pi}{2^{s-k-1}n_k} + \frac{1}{2^{s-k-1}m_k}\right) + \cdots + \left(\frac{\pi}{n_k} + \frac{1}{m_k}\right) \\
&\leq 2\left(\frac{\pi}{n_k} + \frac{1}{m_k}\right).
\end{aligned}$$

Since the final quantity tends to 0 as k tends to infinity, we conclude that the constructed sequence $(N_k)_{k\geq 1}$ is indeed Cauchy. Let N denote the limit of this sequence. Then clearly N is normal, and since each N_k was obtained as the direct sum of a zero matrix and a unitary conjugate of a normal matrix in $\mathbb{M}_{\ell_k}(\mathbb{C})$ within $\pi/n_k + 1/m_k$ of a nilpotent, it follows that

$$dist(N_k, \mathrm{Nil}(\mathbb{M}_{\ell_k}(\mathbb{C}))) \to 0 \text{ as } k \to \infty,$$

and thus $N \in \overline{\mathrm{Nil}(\mathfrak{A})}$. Further, $\|N_k\| \leq 1$ for all k, and consequently $\|N\| \leq 1$ as well. Thus, $\sigma(N) \subseteq \overline{\mathbb{D}}$, and from here it is an easy application of the upper semicontinuity of the spectrum to see that $\sigma(N) = \overline{\mathbb{D}}$.

■

Lemma 5.2.2. *Let $\mathfrak{A}$ be a C^*-algebra and let $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ be such that $\sigma(N) = \overline{\mathbb{D}}$. Then $f(N) \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ whenever $f : \overline{\mathbb{D}} \to \mathbb{C}$ is a continuous function that is analytic on $\mathbb{D}$ and vanishes at 0. Moreover, the above holds if we replace $\overline{\mathrm{Nil}(\mathfrak{A})}$ by $\overline{\mathrm{QNil}(\mathfrak{A})}$.*

Proof. It is well-known that any function f that is continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$ can be approximated uniformly by polynomials on $\overline{\mathbb{D}}$. Further, since f vanishes at 0, we may impose the same requirement on the aforementioned polynomials. Then $f(N) \in \mathrm{Nor}(\mathfrak{A})$, and since $\overline{\mathrm{Nil}(\mathfrak{A})}$ is invariant under these polynomials, it follows that $f(N) \in \overline{\mathrm{Nil}(\mathfrak{A})}$ as well.

■

Theorem 5.2.3. *Let Ω be a non-empty, open, simply connected subset of $\mathbb{C}$ that contains 0, and whose boundary is a simple, closed curve consisting of at least two points. Let $\mathfrak{A}$ be a C^*-algebra and suppose there is an element $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ with $\sigma(N) = \overline{\mathbb{D}}$. Then there exists an element $N_0 \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ with $\sigma(N_0) = \overline{\Omega}$, and further, the above holds if we replace $\overline{\mathrm{Nil}(\mathfrak{A})}$ by $\overline{\mathrm{QNil}(\mathfrak{A})}$.*

Proof. An application of the Riemann mapping theorem yields a bijective holomorphism $f : \mathbb{D} \to \Omega$. This map extends by Carathéodory's theorem to a bijection $g : \overline{\mathbb{D}} \to \overline{\Omega}$ that is continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$. Since $0 \in \Omega$, we can find $a \in \mathbb{D}$ such that $g(a) = 0$. If we define

$$h(z) = \frac{z + a}{\overline{a}z + 1},$$

then is is easy to see that h is a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$, and a bijective holomorphism from $\mathbb{D}$ to $\mathbb{D}$. If we now define $F : \overline{\mathbb{D}} \to \overline{\Omega}$ by $F(z) := g(h(z))$, then

- F is a continuous bijection on $\overline{\mathbb{D}}$,
- F is holomorphic on $\mathbb{D}$, and
- $F(0) = g(h(0)) = g(a) = 0$.

We now simply turn to Lemma 5.2.2 to deduce that $N_0 := F(N) \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$, and since F is bijective, the spectral mapping theorem suggests that $\sigma(N_0) = \overline{\Omega}$.

$\blacksquare$

6 Closed Unitary Orbits of Normal Operators

The problem of determining whether two normal operators are approximately unitarily equivalent has been of interest to mathematicians for many years. Necessary and sufficient conditions have been obtained in certain C^*-algebras, such as $\mathcal{B}(\mathcal{H})$ for a complex, separable, infinite-dimensional Hilbert space $\mathcal{H}$, as well as the Calkin algebra, $\mathcal{Q}(\mathcal{H})$. Later, Skoufranis [20] published an operator theoretic proof of a version of a result due to Dadarlat [5, Theorem 1.7] (see Theorem 7.2.4) which provides a characterization of this phenomenon in the case of a unital, simple, purely infinite C^*-algebra with trivial K_1 group.

The purpose of this section is to explore Skoufranis' approach to Dadarlat's result, culminating in Theorem 6.3.4 (the main result of the section) will see considerable application throughout Section 7 in order to obtain bounds on the distance between normal operator unitary orbits. We commence with an outline of some prerequisites, including a few well-known results of K-theory. To follow, we shall examine this result under the assumption that the normal operators in question have the same connected spectrum, and finally, move to the general case.

6.1 Preliminaries

Although the notions of (approximate) unitary equivalence and similarity have been mentioned in previous sections, it will be helpful to formalize these concepts before proceeding to the heart of the section.

Definition 6.1.1. *Let $\mathfrak{A}$ be a unital C^*-algebra. We let $\mathcal{U}(\mathfrak{A})$ denote the group of unitary elements in $\mathfrak{A}$. Given an element $A \in \mathfrak{A}$, we define the sets*

$$\mathcal{U}(A) := \{UAU^* : U \in \mathcal{U}(\mathfrak{A})\},$$

$$\mathcal{S}(A) := \{SAS^{-1} : S \in \mathrm{GL}(\mathfrak{A})\},$$

*called the **unitary orbit of** A and **similarity orbit of** A, respectively.*

*If $B \in \mathcal{U}(A)$ (resp. $\mathcal{S}(A)$), then we say that B is **unitarily equivalent** to A (resp. **similar** to A) and write $B \sim_u A$ (resp. $B \sim A$). If $B \in \overline{\mathcal{U}(A)}$ then we say that B is **approximately unitarily equivalent** to A and write $B \sim_{au} A$.*

The following two propositions outline some simple properties of the equivalences described in the above definition.

Proposition 6.1.2. *Let $\mathfrak{A}$ be a unital C^*-algebra, and $A, B \in \mathfrak{A}$.*

(1) *$\sim, \sim_u$, and $\sim_{au}$ are equivalence relations.*

(2) *If $A \in \mathrm{Nor}(\mathfrak{A})$ and $A \sim_{au} B$, then B is also normal.*

(3) *If $A \sim B$, then $\sigma(A) = \sigma(B)$.*

(4) *If $B \in \overline{\mathcal{S}(A)}$, then $\sigma(A) \subseteq \sigma(B)$ and $\sigma(A)$ intersects every connected component of $\sigma(B)$.*

(5) *If $A \sim_{au} B$, then $\sigma(A) = \sigma(B)$.*

Proof. Statements $(1)-(3)$ are straightforward to verify. For (4), the fact that $\sigma(A) \subseteq \sigma(B)$ follows from (3) and the upper semicontinuity of the spectrum, and the second claim is immediate from Newburgh's Theorem [2, Theorem 3.4.4]. Lastly, (5) may be quickly deduced from (1) and (4). ■

Proposition 6.1.3. *Let $\mathfrak{A}$ be a unital C^*-algebra. If $A, B \in \mathfrak{A}_{sa}$ and $A \sim B$, then $A \sim_u B$.*

Proof. Let $Z \in \mathrm{GL}(\mathfrak{A})$ be such that $B = ZAZ^{-1}$ and let U denote the unitary operator in the polar decomposition of Z. We have that $BZ = ZA$ and hence

$$Z^*B = (BZ)^* = (ZA)^* = AZ^*.$$

This implies that

$$|Z|^2 A = Z^*ZA = Z^*BZ = AZ^*Z = A|Z|^2,$$

and thus A commutes with $C^*(1, |Z|^2)$ (and in particular, with $|Z|^{-1}$). From this fact it is clear that

$$\begin{aligned} UAU^* &= Z|Z|^{-1}A|Z|^{-1}Z^* \\ &= ZA|Z|^{-2}Z^* \\ &= BZ|Z|^{-2}Z^* = B, \end{aligned}$$

and we conclude that $A \sim_u B$.

■

As a remark, we note that Putnam's generalization of Fuglede's theorem implies that the above proposition also holds in the case where A and B are normal.

The remainder of this subsection serves to examine the K-theory background necessary for the discussion to follow. Whereas the definitions of K_0 and K_1 in a general C^*-algebra (for which the reader is referred to [4] or [17]) are more complicated, in the case of a unital, simple, purely infinite C^*-algebras, the picture is significantly nicer. In fact, we can use the characterizations of Proposition 6.1.5 to "define" K_0 and K_1 for our algebras. We first state the following definition for the *index function* of a normal operator, as this object will be of great significance in the coming exposition.

Definition 6.1.4. *Let $\mathfrak{A}$ be a unital C^*-algebra and $N \in \mathrm{Nor}(\mathfrak{A})$. Since $C^*(N) \cong \mathcal{C}(\sigma(N))$, we obtain a canonical injective $*$-homomorphism from $\mathcal{C}(\sigma(N))$ into $\mathfrak{A}$. This in turn induces a group homomorphism*

$$\Gamma(N) : K_1(\mathcal{C}(\sigma(N))) \to K_1(\mathfrak{A}),$$

called the ***index function*** *of N. For each $\lambda \in \mathbb{C} \setminus \sigma(N)$, we write $\Gamma(N)(\lambda)$ to denote $[N - \lambda I]_1$.*

The following proposition describes the structure of K_0 and K_1 in the setting of a unital, simple, purely infinite C^*-algebra. The proofs of statements (1) and (2) are outlined in [17, Exercise 5.7] and [17, Exercise 8.13], respectively, while the proof of (3) appears in [13].

Proposition 6.1.5. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra.*

(1) *If p, q are two non-zero projections in $\mathfrak{A}$, then $p \sim_0 q$ if and only if $[p]_0 = [q]_0$.*

(2) *If $\mathcal{U}(\mathfrak{A})_0$ denotes the connected component of I in $\mathcal{U}(\mathfrak{A})$, then*

$$K_1(\mathfrak{A}) \cong \mathcal{U}(\mathfrak{A})/\mathcal{U}(\mathfrak{A})_0 \cong \mathrm{GL}(\mathfrak{A})/\mathrm{GL}(\mathfrak{A})_0.$$

(3) *If $N \in \mathrm{Nor}(\mathfrak{A})$ and $\lambda \in \mathbb{C} \setminus \sigma(N)$, then $\Gamma(N)(\lambda)$ describes the connected component of $N - \lambda I$ in $\mathrm{GL}(\mathfrak{A})$.*

As statement (3) in the above proposition indicates, the index function generalizes the notion of index in the Calkin algebra. Moreover, this proposition suggests a reformulation of Theorem 4.1.3 in terms of index. This result can be found in [13], though is stated here for future reference.

Theorem 6.1.6 (Lin)**.** *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, and let $N \in \mathrm{Nor}(\mathfrak{A})$. Then N can be approximated by normal elements with finite spectra if and only if $\Gamma(N)$ is trivial.*

We conclude with the following lemma that will see application in the main result of Section 8. Since its proof requires only the simple facts described above, we present it here.

Lemma 6.1.7. *Let $\mathfrak{A}$ be a unital C^*-algebra and $N_1, N_2 \in \mathrm{Nor}(\mathfrak{A})$ with $N_1 \in \overline{\mathcal{S}(N_2)}$.*

(1) *If $N_2 - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$ for some $\lambda \in \mathbb{C} \setminus \sigma(N_1)$ then $N_1 - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$.*

(2) *If $\mathfrak{A}$ is simple and purely infinite, then $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \sigma(N_1)$.*

Proof. (1) Let $\lambda \in \mathbb{C} \setminus \sigma(N_1)$ be such that $N_2 - \lambda I \in \mathrm{GL}(\mathfrak{A})_0$, and let $\{V_n\}_{n=1}^{\infty}$ be a sequence in $\mathrm{GL}(\mathfrak{A})$ such that

$$\lim_{n\to\infty} \|N_1 - V_n N_2 V_n^{-1}\| = 0.$$

As $\mathrm{GL}(\mathfrak{A})_0$ defines a normal subgroup of $\mathrm{GL}(\mathfrak{A})$, we see that $V_n(N_2 - \lambda I)V_n^{-1} \in \mathrm{GL}(\mathfrak{A})_0$ for all $n \in \mathbb{N}$. Since $\mathrm{GL}(\mathfrak{A})_0$ is relatively closed in $\mathrm{GL}(\mathfrak{A})$ and

$$\lim_{n\to\infty} \|(N_1 - \lambda I) - V_n(N_2 - \lambda I)V_n^{-1}\| = 0,$$

the result is quickly obtained.

(2) Suppose that $\mathfrak{A}$ is simple and purely infinite. Let $\lambda \in \mathbb{C} \setminus \sigma(N_1)$ and let $\{V_n\}_{n=1}^{\infty}$ be as above. Note that since

$$\lim_{n\to\infty} \|(N_1 - \lambda I) - V_n(N_2 - \lambda I)V_n^{-1}\| = 0,$$

it is apparent that $N_1 - \lambda I$ and $V_n(N_2 - \lambda I)V_n^{-1}$ belong to the same connected component of $\mathrm{GL}(\mathfrak{A})$ for sufficiently large values of n. Proposition 6.1.5 now implies that for such values of n, we have

$$\begin{aligned}\Gamma(N_1)(\lambda) = [N_1 - \lambda I]_1 &= \left[V_n(N_2 - \lambda I)V_n^{-1}\right]_1 \\ &= [V_n]_1 [N_2 - \lambda I]_1 \left[V_n^{-1}\right]_1 = [N_2 - \lambda I]_1 = \Gamma(N_2)(\lambda).\end{aligned}$$

This completes the proof.

■

6.2 Normal Operators with the Same Connected Spectrum

With the tools of Section 6.1 in hand, we aim to characterize when two normal operators with the same connected spectrum in a unital, simple, purely infinite C^*-algebra are approximate unitarily equivalent. By considering this case separately, we showcase an important technique that appears in its proof. This key argument will be relied on to prove a number of results in the coming sections, and hence its importance can not be overstated.

Proposition 6.2.1. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathrm{Nor}(\mathfrak{A})$ be such that*

- $\sigma(N_1) = \sigma(N_2)$,
- $\sigma(N_1)$ *is connected, and*
- $\Gamma(N_1)$ *and* $\Gamma(N_2)$ *are trivial.*

Then $N_1 \sim_{au} N_2$.

Proof. Consider first the case that $\sigma(N_1) = \sigma(N_2) = [0,1]$. Let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ so that $1/n < \varepsilon$. By Theorem 6.1.6, there exist elements N_1' and N_2' in $\mathrm{Nor}(\mathfrak{A})$ with finite spectra, and such that $\|N_i - N_i'\| \leq \varepsilon$ for each $i \in \{1,2\}$. Note that by the upper semicontinuity of the spectrum, Lemma 2.2.4, and by applying small perturbations to the elements of $\sigma(N_1')$ and $\sigma(N_2')$, we may assume for each $i \in \{1,2\}$ that

$$\sigma(N_i') = \left\{\frac{j}{n}\right\}_{j=1}^{n}$$

and $\|N_i - N_i'\| \leq 2\varepsilon$. For each $i \in \{1,2\}$ and $j \in \{1,\ldots,n\}$, let $P_j^{(i)}$ denote the spectral projection of N_i' corresponding to $\{j/n\}$, and define $P_0^{(i)} := I - \sum_{j=1}^{n} P_j^{(i)}$. In this case, we obtain two collections $\left\{P_j^{(1)}\right\}_{j=0}^{n}$ and $\left\{P_j^{(2)}\right\}_{j=0}^{n}$ of non-zero, pairwise orthogonal projections such that for each $i \in \{1,2\}$,

$$I = \sum_{j=0}^{n} P_j^{(i)} \ , \text{ and } \quad \left\| N_i - \sum_{j=0}^{n} \frac{j}{n} P_j^{(i)} \right\| \leq 2\varepsilon.$$

As noted prior to the proposition, the following embedding argument will be used a number of times in the coming results, and from here on we shall refer to it as the **interlacing argument**. Recall from Corollary 3.2.6 that every non-zero projection in $\mathfrak{A}$ in infinite. In particular, $P_0^{(1)}$ is infinite, and so Corollary 3.2.10 implies that $P_0^{(1)}$ is Murray–von Neumann equivalent to a proper subprojection of $P_0^{(2)}$. That being said, we can write $P_0^{(2)} = Q_0^{(2)} + R_0^{(2)}$ where $Q_0^{(2)}$ and $R_0^{(2)}$ are non-zero orthogonal projections, and $P_0^{(1)} \sim_0 Q_0^{(2)}$. Similarly, $R_0^{(2)}$ is Murray–von Neumann equivalent to a proper subprojection of $P_1^{(1)}$, and hence we may write $P_1^{(1)} = Q_1^{(1)} + R_1^{(1)}$ where $Q_1^{(1)}$ and $R_1^{(1)}$ are non-zero orthogonal projections, and $R_0^{(2)} \sim_0 Q_1^{(1)}$. By repeating this process a finite number of times, and by defining

$$Q_0^{(1)} := 0\,,\ \ R_0^{(1)} := P_0^{(1)}\,,\ \ Q_n^{(2)} := P_n^{(2)}\,,\ \ R_n^{(2)} := 0\ ,$$

we obtain sets $\left\{Q_j^{(1)}, R_j^{(1)}\right\}_{j=0}^{n}$ and $\left\{Q_j^{(2)}, R_j^{(2)}\right\}_{j=0}^{n}$ of pairwise orthogonal projections such that

- $P_j^{(i)} = Q_j^{(i)} + R_j^{(i)}$ for all $i \in \{1,2\}$ and all $j \in \{0,\ldots,n\}$,
- $R_j^{(2)} \sim_0 Q_{j+1}^{(1)}$ for all $j \in \{0,\ldots,n-1\}$, and
- $R_j^{(1)} \sim_0 Q_j^{(2)}$ for all $j \in \{0,\ldots,n-1\}$.

Using the above information together with the fact that the $P_j^{(i)}$'s sum to I, Proposition 6.1.5 demonstrates that

$$\begin{aligned}\left[R_n^{(1)}\right]_0 &= [I]_0 - \sum_{j=1}^{n}\left[Q_j^{(1)}\right]_0 - \sum_{j=0}^{n-1}\left[R_j^{(1)}\right]_0 \\ &= [I]_0 - \sum_{j=1}^{n}\left[R_{j-1}^{(2)}\right]_0 - \sum_{j=0}^{n-1}\left[Q_j^{(2)}\right]_0 = \left[Q_n^{(2)}\right]_0 .\end{aligned}$$

With this in mind, we can choose a collection $\{V_j\}_{j=0}^{n} \cup \{W_j\}_{j=0}^{n-1}$ of partial isometries such that $V_j^*V_j = R_j^{(1)}$ and $V_jV_j^* = Q_j^{(2)}$ for all $j \in \{0, \ldots, n\}$, and $W_j^*W_j = Q_{j+1}^{(1)}$ and $W_jW_j^* = R_j^{(2)}$ for all $j \in \{0, \ldots, n-1\}$. Using these partial isometries, we define a unitary operator

$$U := \sum_{j=0}^{n} V_j + \sum_{j=0}^{n-1} W_j$$

and note that

$$U^*N_2'U = U^*\left(\sum_{j=0}^{n} \frac{j}{n} P_j^{(2)}\right) U = U^*\left(\sum_{j=0}^{n} \frac{j}{n} Q_j^{(2)} + \sum_{j=0}^{n} \frac{j}{n} R_j^{(2)}\right) U = \sum_{j=0}^{n} \frac{j}{n} R_j^{(1)} + \sum_{j=0}^{n-1} \frac{j}{n} Q_{j+1}^{(1)}.$$

Hence, $\|N_1' - U^*N_2'U\| \leq 1/n < \varepsilon$, and it follows that

$$\|N_1 - U^*N_2U\| \leq \|N_1 - N_1'\| + \|N_1' - U^*N_2'U\| + \|U^*N_2'U - U^*N_2U\| \leq 2\varepsilon + \varepsilon + 2\varepsilon = 5\varepsilon.$$

Therefore, $N_1 \sim_{au} N_2$.

Suppose now that N_1 and N_2 are elements of $\mathrm{Nor}(\mathfrak{A})$ that share the same connected spectrum and have trivial index functions. Let $\varepsilon > 0$, and for each pair $(n, m) \in \mathbb{Z}^2$, let $B_{n,m}$ be defined as in Theorem 4.1.6. We say a box $B_{n,m}$ is *relevant* if $\sigma(N_1) \cap B_{n,m} \neq \emptyset$ and call two boxes *adjacent* if their union is connected. Since we've assumed that $\sigma(N_1)$ is connected, the union of relevant boxes is a connected subset of $\mathbb{C}$. An application of Theorem 6.1.6 yields normal operators M_1 and M_2, both with finite spectra, and such that $\|N_i - M_i\| \leq \varepsilon$ for each $i \in \{1, 2\}$. Again, by the semicontinuity of the spectrum, Lemma 2.2.4, and by making small perturbations to $\sigma(M_1)$ and $\sigma(M_2)$, we may assume that $\sigma(M_1)$ and $\sigma(M_2)$ are precisely the centers of the relevant boxes and $\|N_i - M_i\| \leq 2\varepsilon$ for $i \in \{1, 2\}$.

Let the center of each relevant box define a vertex for a graph G, where two vertices are connected by an edge if and only if their corresponding relevant boxes are adjacent. Consider a spanning tree τ of G, and let λ be an element of $\sigma(M_1)$ that corresponds to a leaf in τ. Let λ' denote the element of $\sigma(M_1)$ corresponding to the vertex in τ that is connected to the aforementioned leaf, and note that the distance between λ and λ' is at most $\sqrt{2}\varepsilon$. Identify this leaf with the spectral projections of M_1 and M_2 corresponding to λ. One may now make use of the interlacing argument from the first case: the spectral projection of M_1 corresponding to λ is Murray–von Neumann equivalent to a proper subprojection of this spectral projection of M_2. Whatever remains is, in turn, Murray–von Neumann equivalent to a proper subprojection of the spectral projection of M_1 corresponding to λ'. Our leaf may now be removed from τ to obtain a small tree, and the above analysis can be applied once again.

After a finite number of repetitions we arrive at the trivial tree, and K-theory arguments similar to those above can be used to show that the remaining projections are Murray–von Neumann equivalent. As before, we can use the partial isometries implementing the equivalences to construct a unitary, U, such that $\|M_1 - U^*M_2U\| \leq \sqrt{2}\varepsilon$. This fact now indicates that

$$\|N_1 - U^*N_2U\| \leq \|N_1 - M_1\| + \|M_1 - U^*M_2U\| + \|U^*M_2U - U^*N_2U\| \leq 2\varepsilon + \sqrt{2}\varepsilon + 2\varepsilon < 6\varepsilon,$$

and thus $N_1 \sim_{au} N_2$.

■

6.3 Normal Operators with Equivalent Common Spectral Projections

Having determined when two normal operators with the same connected spectrum are approximately unitarily equivalent (and more importantly, having explored the interlacing argument applied in its proof), we are now nearly prepared to consider the general case. A few preliminary lemmas will be first be established.

Lemma 6.3.1. *Let $\mathfrak{A}$ be a unital C^*-algebra and $P, Q \in \mathfrak{A}$ be projections. If there exists an element $V \in \mathrm{GL}(\mathfrak{A})_0$ such that*

$$\|Q - VPV^{-1}\| < \frac{1}{2},$$

then P and Q are Murray–von Neumann equivalent.

Proof. Define $P_0 := VPV^{-1}$ and $Z := P_0Q + (I - P_0)(I - Q)$, and note that

$$\begin{aligned} \|Z - I\| &= \|(P_0Q + (I - P_0)(I - Q)) - (Q - (I - Q))\\ &\leq \|(P_0 - I)Q\| + \|((I - P_0) - I)(I - Q)\| \\ &= \|(P_0 - Q)Q\| + \|((I - P_0) - (I - Q))(I - Q)\| \\ &\leq \underbrace{\|P_0 - Q\|}_{<\frac{1}{2}} + \underbrace{\|Q - P_0\|}_{<\frac{1}{2}} < 1 \end{aligned}$$

Hence, Z defines an element of $\mathrm{GL}(\mathfrak{A})$, and so if we let $Z = U|Z|$ be the polar decomposition of Z, then U is unitary in $\mathfrak{A}$ and $UQU^* = P_0$. To see this, note that by definition of Z, it is evident that $ZQ = P_0Q = P_0Z$ and

$$Z^*Z = QP_0Q + (I - Q)(I - P_0)(I - Q).$$

Combining these results, it readily follows that $QZ^*Z = QP_0Q = Z^*ZQ$, and hence Q commutes with $C^*(Z^*Z)$. In particular, Q commutes with $|Z|^{-1}$ and we obtain

$$UQU^* = Z|Z|^{-1}Q|Z|^{-1}Z^* = ZQ|Z|^{-2}Z^* = P_0Z|Z|^{-2}Z^* = P_0.$$

Thus, $Q = (U^*V)P(U^*V)^{-1}$, at which point an application of Proposition 6.1.3 shows that $P \sim_u Q$. From here it is straightforward to verify that $P \sim_0 Q$. ■

The following lemma demonstrates sufficient conditions for a function to (in some sense) preserve the elements of a closed unitary orbit or closed similarity orbit of a given operator.

Lemma 6.3.2. *Let $\mathfrak{A}$ be a unital C^*-algebra and let $A, B \in \mathfrak{A}$. Let $f : \mathbb{C} \to \mathbb{C}$ be a function that is analytic on an open neighbourhood U of $\sigma(A) \cup \sigma(B)$.*

(1) *If $A \in \overline{\mathcal{S}(B)}$, then $f(A) \in \overline{\mathcal{S}(f(B))}$.*

(2) *If $A \sim_{au} B$, then $f(A) \sim_{au} f(B)$.*

Proof. Let $\{V_n\}_{n=1}^\infty$ be a sequence in $\mathrm{GL}(\mathfrak{A})$ such that $\lim_{n\to\infty} \|A - V_nBV_n^{-1}\| = 0$, and consider any compact rectifiable curve γ inside U such that

- $(\sigma(A) \cup \sigma(B)) \cap \gamma = \emptyset$,
- $\mathrm{Ind}_\gamma(z) \in \{0, 1\}$ for all $z \in \mathbb{C} \setminus \gamma$,

- $\mathrm{Ind}_\gamma(z) = 1$ for all $z \in \sigma(A) \cup \sigma(B)$, and
- $\{z \in \mathbb{C} : \mathrm{Ind}_\gamma(z) \neq 0\} \subseteq U$.

By the second resolvent equation and the fact that $V_n(zI - B)^{-1}V_n^{-1} = (zI - V_nBV_n^{-1})^{-1}$ for every $n \in \mathbb{N}$, it follows that

$$\begin{aligned} f(A) - V_nf(B)V_n^{-1} &= \frac{1}{2\pi i}\int_\gamma f(z)\left[(zI-A)^{-1} - V_n(zI-B)^{-1}V_n^{-1}\right]dz \\ &= \frac{1}{2\pi i}\int_\gamma f(z)\left[(zI-A)^{-1} - (zI - V_nBV_n^{-1})^{-1}\right]dz \\ &= \frac{1}{2\pi i}\int_\gamma f(z)(zI-A)^{-1}(A - V_nBV_n^{-1})(zI - V_nBV_n^{-1})^{-1}dz. \end{aligned}$$

We may now use this identity to obtain the estimate

$$\|f(A) - V_nf(B)V_n^{-1}\| \leq \frac{\ell(\gamma)\|A - V_nBV_n^{-1}\|}{2\pi}\sup_{z\in\gamma}|f(z)|\|(zI-A)^{-1}\|\|(zI - V_nBV_n^{-1})^{-1}\|, \qquad (*)$$

where $\ell(\gamma)$ denotes the length of the curve γ. If we define $T := (zI - A)^{-1}(A - V_nBV_n^{-1})$, then by choosing $n \in \mathbb{N}$ large enough so that $\|T\| < 1$, it follows that

$$(I+T)^{-1} = I - T + T^2 - T^3 + \cdots$$

is absolutely convergent, and

$$\|(I+T)^{-1}\| \leq \frac{1}{1 - \|T\|}.$$

Again, we may turn to the second resolvent equation to see that

$$(zI - V_nBV_n^{-1})^{-1} = (I + (zI-A)^{-1}(A - V_nBV_n^{-1}))^{-1}(zI-A)^{-1},$$

and combining with the above calculations, it quickly follows that for n sufficiently large,

$$\|(zI - V_nBV_n^{-1})^{-1}\| \leq \frac{\|(zI-A)^{-1}\|}{1 - \|A - V_nBV_n^{-1}\|\|(zI-A)^{-1}\|}.$$

Returning to the inequality given by $(*)$, the above implies that

$$\|f(A) - V_nf(B)V_n^{-1}\| \leq \frac{\ell(\gamma)\|A - V_nBV_n^{-1}\|}{2\pi}\sup_{z\in\gamma}|f(z)|\frac{\|(zI-A)^{-1}\|}{1 - \|A - V_nBV_n^{-1}\|\|(zI-A)^{-1}\|}.$$

Since we are considering the supremum of a continuous function of z over a compact set γ, it is clear that this supremum is finite, and since $\|A - V_nBV_n^{-1}\|$ tends to zero as n becomes large, we conclude that so too does $\|f(A) - V_nf(B)V_n^{-1}\|$. This proves that $f(A) \in \overline{\mathcal{S}(f(B))}$. Note that the case where $A \sim_{au} B$ follows in a similar way upon selecting the V_n's to be unitary.

■

If f is a function satisfying the assumptions of Lemma 6.3.2 for normal operators A and B, and is such that $f(A)$ and $f(B)$ are projections in $\mathfrak{A}$, then one may ask if $f(A)$ and $f(B)$ are Murray–von Neumann equivalent. This motivates the following definition:

Definition 6.3.3. *Let $\mathfrak{A}$ be a unital C^*-algebra and $N_1, N_2 \in \mathrm{Nor}(\mathfrak{A})$. We say that N_1 and N_2 have* ***equivalent common spectral projections*** *if for every function $f : \mathbb{C} \to \mathbb{C}$ that is analytic on an open set U of $\sigma(N_1) \cup \sigma(N_2)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N_1)$ and $f(N_2)$ are Murray–von Neumann equivalent.*

We are now in a position to tackle the main result of Section 6.

Theorem 6.3.4. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathrm{Nor}(\mathfrak{A})$ be such that*

(1) $\sigma(N_1) = \sigma(N_2)$,

(2) $\Gamma(N_1)$ *and* $\Gamma(N_2)$ *are trivial, and*

(3) N_1 *and* N_2 *have equivalent common spectral projections.*

Then $N_1 \sim_{au} N_2$.

Proof. Let $\varepsilon > 0$. For each pair $(n, m) \in \mathbb{Z}^2$, define $B_{n,m}$ as in Theorem 4.1.6, and call a box $B_{n,m}$ *relevant* if $B_{n,m} \cap \sigma(N_1) \neq \emptyset$. If we let K denote the union of the relevant boxes, then K need not be connected but the compactness of $\sigma(N_1)$ implies that K has finitely many connected components, say $L_1, \ldots, L_k$. For each $i \in \{1, \ldots, k\}$, define f_i to be the characteristic function on L_i, and notice that since N_1 and N_2 were assumed to have equivalent common spectral projections, it is true that $f_i(N_1) \sim_0 f_i(N_2)$ for all i.

The assumption that the index functions for N_1 and N_2 are trivial allows us to invoke Theorem 6.1.6 and obtain normal elements M_1 and M_2 in $\mathfrak{A}$, both with finite spectra, and such that $\|M_j - N_j\| \leq \varepsilon$ for each $j \in \{1, 2\}$. By the semicontinuity of the spectrum, Lemma 2.2.4, and by making small perturbations to $\sigma(M_1)$ and $\sigma(M_2)$, we may assume for each $j \in \{1, 2\}$ that

- $\sigma(M_j)$ is precisely the centers of the relevant boxes,
- $\sigma(M_j) \cap B_{n,m} \neq \emptyset$ whenever $B_{n,m}$ is relevant, and
- $\|N_j - M_j\| \leq 2\varepsilon$.

Further, since $\mathrm{dist}(L_i, L_\ell) \geq \varepsilon$ for all $i, \ell \in \{1, \ldots, k\}$ with $i \neq \ell$, it follows that each f_i is continuous on K, and hence we may assume that

$$\|f_i(N_j) - f_i(M_j)\| < \frac{1}{2}$$

for all $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$. Lemma 6.3.1 now guarantees that $f_i(N_j) \sim_0 f_i(M_j)$ for all i and j, and hence

$$f_i(M_1) \sim_0 f_i(N_1) \sim_0 f_i(N_2) \sim_0 f_i(M_2)$$

for every $i \in \{1, \ldots, k\}$.

Our attention now turns to the interlacing argument outlined in Proposition 6.2.1. For each $i \in \{1, \ldots, k\}$, we can form a tree in the connected component L_i and then embed the spectral projection of M_1 corresponding to a fixed leaf of this tree under the spectral projection of M_2 corresponding to the same leaf. We then embed what remains of the spectral projection of M_2 under the spectral projection of M_1 corresponding to the vertex adjacent to the aforementioned leaf. The leaf is then removed and the analysis is repeated on the now smaller tree. Suppose that for a fixed value of $\ell \in \{1, \ldots, k\}$, the above process is carried out and we arrive at two sets, $\left\{Q_i^{(1)}, R_i^{(1)}\right\}_{i=0}^n$ and $\left\{Q_i^{(2)}, R_i^{(2)}\right\}_{i=0}^n$, of pairwise orthogonal projections with the following properties:

- $R_i^{(1)} \sim_0 Q_i^{(2)}$ for all $i \in \{0, \ldots, n-1\}$,
- $R_i^{(2)} \sim_0 Q_{i+1}^{(1)}$ for all $i \in \{0, \ldots, n-1\}$, and
- $\displaystyle\sum_{i=0}^{n} R_i^{(j)} + \sum_{i=0}^{n} Q_i^{(j)} = f_\ell(M_j)$ for each $j \in \{1, 2\}$.

By arguments analogous to those in Proposition 6.2.1, the equivalence of $f_\ell(M_1)$ and $f_\ell(M_2)$ together with the above observations now imply that $\left[R_n^{(1)}\right]_0 = \left[Q_n^{(2)}\right]_0$, and hence $R_n^{(1)} \sim_0 Q_n^{(2)}$. Out of these equivalences comes a collection $\{V_i\}_{i=1}^{k}$ of partial isometries such that

$$V_i^* V_i = f_i(M_1)\ ,\ \ V_i V_i^* = f_i(M_2)\ ,\ \text{ and } \ \|M_1 f_i(M_1) - V_i^* M_2 f_i(M_2) V_i\| \leq \sqrt{2}\varepsilon.$$

By defining U to be the sum of these partial isometries, we may use the fact that $\sum_{i=1}^{n} f_i(M_j) = I$ for each $j \in \{1, 2\}$ to deduce that U is unitary satisfying $\|M_1 - U^* M_2 U\| \leq \sqrt{2}\varepsilon$. Consequently,

$$\|N_1 - U^* N_2 U\| \leq \|N_1 - M_1\| + \|M_1 - U^* M_2 U\| + \|U^* M_2 U - U^* N_2 U\| \leq 2\varepsilon + \sqrt{2}\varepsilon + 2\varepsilon < 6\varepsilon,$$

and hence $N_1 \sim_{au} N_2$, as desired.

■

We shall state an extension of this result in case of normal operators with non-trivial index functions in Section 7. For now, the following corollary describes a situation in which a converse for Theorem 6.3.4 is obtained.

Corollary 6.3.5. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra with trivial K_1 group, and let $N_1, N_2 \in \mathrm{Nor}(\mathfrak{A})$. Then $N_1 \sim_{au} N_2$ if and only if*

- *$\sigma(N_1) = \sigma(N_2)$, and*
- *N_1 and N_2 have equivalent common spectral projections.*

Proof. Suppose first that $N_1 \sim_{au} N_2$. Then $\sigma(N_1) = \sigma(N_2)$, and if $f : \mathbb{C} \to \mathbb{C}$ is a function that is analytic on $\sigma(N_1) \cup \sigma(N_2)$, then Lemma 6.3.2 may be applied to show that $f(N_1) \sim_{au} f(N_2)$. It is then evident that $f(N_1) \sim_0 f(N_2)$ by Lemma 6.3.1, and hence N_1 and N_2 have equivalent common spectral projections.

Conversely, if N_1 and N_2 have identical spectra and equivalent common spectral projections, then the fact that $K_1(\mathfrak{A})$ is trivial ensures that N_1 and N_2 have trivial index functions. From here we may apply Theorem 6.3.4 to conclude that $N_1 \sim_{au} N_2$.

■

7 Distance Between Unitary Orbits of Normal Operators

We now turn our attention to obtaining bounds on the distance between unitary orbits of normal operators. This distance is somewhat well understood in the case of bounded operators acting on a complex, infinite-dimensional Hilbert space (see [7] and [8] for example), as well as in the Calkin algebra [6].

Our focus will be the setting of a unital, simple, purely infinite C^*-algebra. It is in this setting that Skoufranis [20] was able to determine the distance between unitary orbits of any two normal operators satisfying certain simple conditions. This result is given by Theorem 7.2.5. In order to construct a proof of Skoufranis' result, we shall first consider the case of normal operators whose index functions are trivial. It is here that we will see much application from Section 6, as such normal operators were of exclusive interest. The case of normal operators with possibly non-trivial index functions will require the full power of Dadarlat's result [5, Theorem 1.7] (see Theorem 7.2.4), but will be used to exhibit an important characterization of normal operator similarity orbits in Section 8.

7.1 Normal Operators with Trivial Index Function

Our analysis begins with the case of normal operators with trivial index functions. As we shall see, this distance is intimately related to a certain notion of distance between the spectra of the given normal operators, known as the *Hausdorff distance*. This is defined in the following way:

Definition 7.1.1. *Let $X, Y \subseteq \mathbb{C}$. The **Hausdorff distance** between X and Y is defined by*

$$d_H(X,Y) := \max\left\{ \sup_{x\in X} \operatorname{dist}(x,Y)\ ,\ \sup_{y\in Y} \operatorname{dist}(y,X) \right\}.$$

We note that this defines a metric on the set of compact subsets of $\mathbb{C}$.

The following proposition is an adaptation of [6, Proposition 1.2] and indicates that the Hausdorff distance between the spectra of two normal operators in *any* unital C^*-algebra is a lower bound for the distance between their unitary orbits. We omit the proof.

Proposition 7.1.2. *Let $\mathfrak{A}$ be a unital C^*-algebra and $N_1, N_2 \in \operatorname{Nor}(\mathfrak{A})$. Then*

$$\operatorname{dist}(\mathcal{U}(N_1),\mathcal{U}(N_2)) \geq d_H(\sigma(N_1),\sigma(N_2)).$$

As it happens, one may achieve equality in Proposition 7.1.2 by imposing certain conditions on the normal operators in question. These conditions are outlined formally in the following lemma, wherein they are proven sufficient by means of the interlacing argument of Proposition 6.2.1.

Lemma 7.1.3. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \operatorname{Nor}(\mathfrak{A})$ be such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. If $\sigma(N_1)$ is connected, then*

$$\operatorname{dist}(\mathcal{U}(N_1),\mathcal{U}(N_2)) = d_H(\sigma(N_1),\sigma(N_2)).$$

Proof. Of course we have that

$$\operatorname{dist}(\mathcal{U}(N_1),\mathcal{U}(N_2)) \geq d_H(\sigma(N_1),\sigma(N_2))$$

by Proposition 7.1.2. To see that the reverse inequality holds, let $\varepsilon > 0$ and for each pair $(n,m) \in \mathbb{Z}^2$, define the set $B_{n,m}$ as in Theorem 4.1.6. For each $j \in \{1,2\}$, we say that a box $B_{n,m}$ is N_j*-relevant* if $\sigma(N_j) \cap B_{n,m} \neq \emptyset$. We apply Theorem 6.1.6 to obtain normal operators $M_1, M_2 \in \mathfrak{A}$ that approximate N_1 and N_2, respectively, and such that $\sigma(M_1)$ and $\sigma(M_2)$ are finite. By familiar arguments (see Proposition 6.2.1, for example) we may assume that $\sigma(M_j)$ is exactly the centers of the N_j-relevant boxes, and $\|N_j - M_j\| \leq 2\varepsilon$ for each $j \in \{1,2\}$. Given $j \in \{1,2\}$ and an element $\lambda \in \sigma(M_j)$, let $P_\lambda^{(j)}$ denote the spectral projection of M_j corresponding to λ.

We will now construct a graph from the spectra of M_1 and M_2, as was done in the proof of Proposition 6.2.1. Fix $\lambda \in \sigma(M_i)$ where $i \in \{1,2\}$, and let $j \in \{1,2\} \setminus \{i\}$. For every $\mu \in \sigma(M_j)$ with

$$|\lambda - \mu| \leq d_H(\sigma(N_1), \sigma(N_2)) + \sqrt{2}\varepsilon,$$

add an edge from μ to λ, and from μ to the center of any N_i-relevant box that is adjacent to the one containing λ. Observe that at least one such μ always exists. To see this, note that if we select μ so that $|\lambda - \mu|$ is a minimum, then

$$|\lambda - \mu| = \operatorname{dist}(\lambda, \sigma(M_2)) \leq d_H(\sigma(M_1), \sigma(M_2)).$$

Further, it is easy to see from the above construction that

$$\begin{aligned} d_H(\sigma(M_1), \sigma(M_2)) &= \max\left\{ \sup_{\lambda \in \sigma(M_1)} \operatorname{dist}(\lambda, \sigma(M_2)) \ , \ \sup_{\mu \in \sigma(M_2)} \operatorname{dist}(\mu, \sigma(M_1)) \right\} \\ &\leq \max\left\{ \sup_{\lambda \in \sigma(N_1)} \operatorname{dist}(\lambda, \sigma(M_2)) + \frac{\sqrt{2}}{2}\varepsilon \ , \ \sup_{\mu \in \sigma(N_2)} \operatorname{dist}(\mu, \sigma(M_1)) + \frac{\sqrt{2}}{2}\varepsilon \right\} \\ &\leq \max\left\{ \sup_{\lambda \in \sigma(N_1)} \operatorname{dist}(\lambda, \sigma(N_2)) + \sqrt{2}\varepsilon \ , \ \sup_{\mu \in \sigma(N_2)} \operatorname{dist}(\mu, \sigma(N_1)) + \sqrt{2}\varepsilon \right\}. \end{aligned}$$

Since this final term is exactly $d_H(\sigma(N_1), \sigma(N_2)) + \sqrt{2}\varepsilon$, it is clear that this choice of μ will give rise to an edge.

The result of this construction is a bipartite graph G, and

$$|\lambda - \mu| \leq d_H(\sigma(N_1), \sigma(N_2)) + 2\sqrt{2}\varepsilon$$

whenever $\lambda \in \sigma(M_1)$ and $\mu \in \sigma(M_2)$ are connected by an edge in G. We claim that G is, in fact, a connected graph. Indeed, since every vertex in G has at least one edge, it suffices to show that for any two distinct elements $\lambda, \mu \in \sigma(M_1)$, there exists a path in G connecting λ to μ. By assumption, $\sigma(N_1)$ is connected, and hence so too is the union of the N_1-relevant boxes. Thus, there is a chain

$$\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu,$$

where $\lambda_{\ell-1}$ and λ_ℓ are centers of adjacent N_1-relevant boxes for each $\ell \in \{1, \ldots, k\}$. But for any fixed such ℓ there exists an element $\mu_\ell \in \sigma(M_2)$ that is connected to λ_ℓ, and hence also to $\lambda_{\ell-1}$ by construction. This defines a path from $\lambda_{\ell-1}$ to λ_ℓ. By adjoining these paths, we obtain a path from λ to μ, and hence G is connected.

Once again, we will mimic the interlacing argument of Proposition 6.2.1. Since G is connected, we may apply Lemma 4.1.5 to find $j \in \{1,2\}$ and $\lambda \in \sigma(M_j)$ such that G is still connected upon removing λ. Let $i \in \{1,2\} \setminus \{j\}$ and suppose that $\mu \in \sigma(M_i)$ is a vertex in G that is connected to λ. In this case, we must have that

$$|\lambda - \mu| \leq d_H(\sigma(N_1), \sigma(N_2)) + 2\sqrt{2}\varepsilon.$$

Since $\mathfrak{A}$ is simple and purely infinite, the projection $P_\lambda^{(j)}$ is Murray–von Neumann equivalent to a proper subprojection $Q_\mu^{(i)}$ of $P_\mu^{(i)}$. Hence, we may write $P_\mu^{(i)} = Q_\mu^{(i)} + R_\mu^{(i)}$, remove λ from G, and replace the projection $P_\lambda^{(j)}$ by $R_\mu^{(i)}$. This produces a smaller connected bipartite graph, and hence the process may be repeated a finite number of times until only two vertices remain. By construction, one of the projections corresponding to these vertices will be a non-zero subprojection of a spectral projection of M_1, and the other will be a non-zero subprojection of a spectral projection of M_2. As in the proof of Proposition 6.2.1, K-theory implies that these remaining projections must be Murray–von Neumann equivalent.

The equivalences described above will generate a collection of partial isometries, the sum of which defines a unitary $U \in \mathfrak{A}$ that satisfies

$$\|M_1 - U^* M_2 U\| \leq d_H(\sigma(M_1), \sigma(M_2)) \leq 2\sqrt{2}\varepsilon + d_H(\sigma(N_1), \sigma(N_2)).$$

Finally, the existence of such a unitary implies that

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq \|N_1 - U^* N_2 U\| \leq (4 + 2\sqrt{2})\varepsilon + d_H(\sigma(N_1), \sigma(N_2)),$$

thereby completing the proof.

■

We will now remove the assumption that either normal operator has connected spectrum. It turns out that a similar result is within reach if the connected components of $\sigma(N_1)$ and $\sigma(N_2)$ can be paired in such a way that for each pair, the indicator functions for N_1 and N_2 on these components are Murray–von Neumann equivalent. It is then that we may apply the previous analysis within each pair of components and obtain following consequence. Note that the symbol "$\sqcup$" will be used to indicate a disjoint union of sets.

Corollary 7.1.4. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathrm{Nor}(\mathfrak{A})$ be such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. Suppose there exists $n \in \mathbb{N}$ such that for each $j \in \{1, 2\}$, we have*

$$\sigma(N_j) = \bigsqcup_{i=1}^{n} K_i^{(j)}$$

where $K_i^{(j)}$ is a compact, connected set for every $i \in \{1, \ldots, n\}$. Let $\chi_i^{(j)}$ denote the characteristic function of $K_i^{(j)}$ for each $j \in \{1, 2\}$ and $i \in \{1, \ldots, n\}$. If $\chi_i^{(1)}(N_1) \sim_0 \chi_i^{(2)}(N_2)$ for all $i \in \{1, \ldots, n\}$, then

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq \max_{1 \leq i \leq n} d_H\left(K_i^{(1)}, K_i^{(2)}\right).$$

Proof. Let $\varepsilon > 0$. We simply apply the arguments from Lemma 7.1.3 within each pair $\left(K_i^{(1)}, K_i^{(2)}\right)$, where $i \in \{1, \ldots, n\}$. That is, we first apply Theorem 6.1.6 to approximate N_1 an N_2 by normal operators, M_1 and M_2, each with finite spectrum. The equivalence of $\chi_i^{(1)}(N_1)$ and $\chi_i^{(2)}(N_2)$ permits the assumption that $\chi_i^{(1)}(M_1) \sim_0 \chi_i^{(2)}(M_2)$ for every $i \in \{1, \ldots, n\}$ by reasoning akin to that of Theorem 6.3.4. This allows one to perform the same K-theoretic arguments of Lemma 7.1.3 and obtain a collection $\{V_1, \ldots, V_n\}$ of partial isometries satisfying $V_i^* V_i = \chi_i^{(1)}(N_1)$, $V_i V_i^* = \chi_i^{(2)}(N_2)$, and

$$\|N_1 \chi_i^{(1)}(N_1) - V_i^* N_2 \chi_i^{(2)}(N_2) V_i\| < \varepsilon + d_H\left(K_i^{(1)}, K_i^{(2)}\right)$$

for all $i \in \{1, \dots, n\}$. Upon considering the sum of such partial isometries, one arrives at a unitary $U \in \mathfrak{A}$ satisfying

$$\|N_1 - U^* N_2 U\| < \varepsilon + \max_{1 \le i \le n} d_H \left(K_i^{(1)}, K_i^{(2)} \right),$$

and hence the result follows.

■

Given the assumptions of the previous corollary, it is not surprising to learn that one may obtain a similar bound in the case of normal operators with trivial index functions and equivalent common spectral projections. This situation is described in the following corollary, which is generalized to an interesting result at the end of this section, and will play a key role in proving the main theorem of Section 8. This should be viewed as the analogue of the corresponding result for normal operators whose spectra and essential spectra coincide, as proven by Davidson in [7].

Corollary 7.1.5. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathrm{Nor}(\mathfrak{A})$ be such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. If N_1 and N_2 have equivalent common spectral projections, then*

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. Let $\varepsilon > 0$ and let M_1 and M_2 be the normal operators obtained in the proof of Lemma 7.1.3. We may argue as in Theorem 6.3.4 and assume that for each $j \in \{1, 2\}$, the projections $\chi_K(N_j)$ and $\chi_K(M_j)$ are Murray–von Neumann equivalent for every connected component K of the union of the relevant N_j-boxes.

Let G denote the bipartite graph constructed in Lemma 7.1.3, and note that because $\sigma(N_1)$ is not assumed to be connected, the graph G may not be connected. However, we will argue that the analysis of Lemma 7.1.3 can be applied within each connected component of G to reach the desired result. To see this, let G_0 be a connected component of G, and for each $j \in \{1, 2\}$, let K_j denote the union of the N_j-relevant boxes that contain vertices of G_0. By defining $K := K_1 \cup K_2$, it is easy to see that for each $j \in \{1, 2\}$, the distance between K and any other N_j-relevant box is at least ε. This implies χ_K defines a continuous function on $\sigma(N_1) \cup \sigma(N_2)$, and since N_1 and N_2 were assumed to have equivalent common spectral projections, it follows that $\chi_K(N_1) \sim_0 \chi_K(N_2)$. Combining with the remarks of the first paragraph, we obtain

$$\chi_K(M_1) \sim_0 \chi_K(N_1) \sim_0 \chi_K(N_2) \sim_0 \chi_K(M_2),$$

allowing the K-theoretic arguments of Lemma 7.1.3 to proceed as before. Each connected component of G therefore gives rise to a finite collection of partial isometries, and since there are only finitely many components, we may sum these partial isometries to arrive at a unitary U satisfying

$$\|N_1 - U^* N_2 U\| \le (4 + 2\sqrt{2})\varepsilon + d_H(\sigma(N_1), \sigma(N_2)).$$

The result is now immediate.

■

7.2 Normal Operators with Non-Trivial Index Function

The problem of computing the distance between unitary orbits of normal operators becomes significantly more challenging without the assumption of trivial index functions. As an example, we remark that this problem is not complete even in the Calkin algebra. In this section, we aim to remedy this situation by constructing a copy of the 2^∞-UHF C^*-algebra within a certain corner of our unital, simple, purely infinite C^*-algebra. The fact that the invertible elements in this UHF C^*-algebra form a connected set will allow Theorem 1.7 in [5] to, in essence, reduce the problem to the setting of Corollary 7.1.5.

Lemma 7.2.1. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, and let $V \in \mathfrak{A}$ be a proper isometry. If we define $P := VV^*$, then there is a unital embedding of the 2^∞-UHF C^*-algebra*

$$\mathfrak{B} := \overline{\bigcup_{\ell \geq 1} \mathbb{M}_{2^\ell}(\mathbb{C})}$$

into the corner $(I-P)\mathfrak{A}(I-P)$ such that $[Q]_0 = 0$ in $\mathfrak{A}$ for every projection $Q \in \mathfrak{B}$.

Proof. Define $P_0 := I - P$ and note that since $\mathfrak{A}$ is purely infinite, Corollary 3.2.6 implies that P_0 is Murray–von Neumann equivalent to a proper subprojection of itself. Call this projection P_1 and let $P_2 := P_0 - P_1$. The isometry V implies that $[P]_0 = [I]_0$ by Proposition 6.1.5, and hence $[P_0]_0 = [I-P]_0 = 0$. That being said, we see that $[P_1]_0 = [P_0]_0 = 0$, and

$$[P_2]_0 = [P_1]_0 + [P_2]_0 = [P_1 + P_2]_0 = [P_0]_0 ,$$

so $P_0 \sim_0 P_1 \sim_0 P_2$ in $\mathfrak{A}$. If we let $S \in \mathfrak{A}$ be such that $SS^* = P_1$ and $S^*S = P_2$, then from the fact that $P_1, P_2 \in P_0\mathfrak{A}P_0$, it follows that $S \in P_0\mathfrak{A}P_0$ as well. This shows that $P_1 \sim_0 P_2$ in $P_0\mathfrak{A}P_0$, and hence we may choose elements $V_1, V_2 \in P_0\mathfrak{A}P_0$ such that $V_i^*V_i = P_0$ and $V_iV_i^* = P_i$ for each $i \in \{1,2\}$.

For each $\ell \in \mathbb{N}$, define

$$\mathfrak{B}_\ell := * - \text{alg}\left(\left\{V_{i_1}\cdots V_{i_\ell}V_{j_\ell}^*\cdots V_{j_1}^* : i_k, j_k \in \{1,2\} \text{ for all } k \in \{1,\ldots,\ell\}\right\}\right).$$

Note that $\mathfrak{B}_\ell$ defines a C^*-subalgebra of $P_0\mathfrak{A}P_0$ that contains P_0, $\mathfrak{B}_\ell \subseteq \mathfrak{B}_{\ell+1}$ for all $\ell \in \mathbb{N}$, and $\mathfrak{B}_\ell \cong \mathbb{M}_{2^\ell}(\mathbb{C})$ (which can be obtained by mapping the generators of $\mathfrak{B}_\ell$ to the matrix units of $\mathbb{M}_{2^\ell}(\mathbb{C})$ in the obvious way). From this isomorphism it is easy to see that

$$\mathfrak{B} := \overline{\bigcup_{\ell \in \mathbb{N}} \mathfrak{B}_\ell}$$

defines a copy of the 2^∞-UHF C^*-algebra in $P_0\mathfrak{A}P_0$.

Suppose that R is a rank-1 projection in $\mathfrak{B}_\ell$. Since any two rank-1 projections in $\mathfrak{B}_\ell$ are Murray–von Neumann equivalent, it follows that every rank-1 projection in $\mathfrak{B}_\ell$ is equivalent to $V_1^\ell(V_1^*)^\ell$ (as this element corresponds to the matrix unit $E_{1,1} \in \mathbb{M}_{2^\ell}(\mathbb{C})$). But $V_1^\ell(V_1^*)^\ell$ is Murray–von Neumann equivalent to P_0 in $\mathfrak{A}$ by construction, and thus $[R]_0 = [P_0]_0 = 0$ in $\mathfrak{A}$. With this in mind, given a projection $Q \in \mathfrak{B}$, we may choose $\ell \in \mathbb{N}$ and a projection $Q_0 \in \mathfrak{B}_\ell$ such that

$$\|Q - Q_0\| < 1/2,$$

and by Lemma 6.3.1 we see that $Q \sim_0 Q_0$ in $\mathfrak{A}$. Since Q_0 can be expressed as a sum of rank-1 projections in $\mathfrak{B}_\ell$, the above analysis demonstrates that $[Q_0]_0 = 0$ in $\mathfrak{A}$ and therefore $[Q]_0 = 0$ in $\mathfrak{A}$ as well.

∎

With the above construction complete, we now demonstrate that any compact subset of $\mathbb{C}$ can be seen as the spectrum of a normal operator in this copy of the 2^∞-UHF C^*-algebra.

Lemma 7.2.2. *Let $X \subseteq \mathbb{C}$ be a compact set, and let $\mathfrak{B}$ denote the 2^∞-UHF C^*-algebra as in Lemma 7.2.1. Then there exists an element $N \in \mathrm{Nor}(\mathfrak{B})$ such that $\sigma(N) = X$.*

Proof. For each $n \in \mathbb{N}$, let $X_n = \{x_{j,n}\}_{j=1}^{k_n}$ be a 2^{-n}-net in X and let $\ell_1 \in \mathbb{N}$ be such that $k_1 \leq 2^{\ell_1}$. Then

$$N_1 := \mathrm{diag}(x_{1,1}, \ldots, x_{k_1,1}, \underbrace{x_{k_1,1}, \ldots, x_{k_1,1}}_{2^{\ell_1}-k_1})$$

defines a normal element of $\mathbb{M}_{2^{\ell_1}}(\mathbb{C})$ with $\sigma(N_1) = X_1$. We now let $\ell_2 = \ell_1(1+k_2)$. For every $j \in \{1, \ldots, k_2\}$, choose $i_j \in \{1, \ldots, k_1\}$ such that $|x_{j,2} - x_{i_j,1}| < 2^{-1}$ (which is possible as X_1 is a 2^{-1}-net for X) and let $N_{j,2}$ denote the matrix N_1 where the first occurrence of $x_{i_j,1}$ is replaced by $x_{j,2}$. Now by defining

$$N_2 := N_1 \oplus N_{1,2} \oplus N_{2,2} \oplus \cdots \oplus N_{k_2,2},$$

we see that N_2 is a normal element of $\mathbb{M}_{2^{\ell_2}}(\mathbb{C})$ with $\sigma(N_2) = X_1 \cup X_2$, and since $\|N_1 - N_{j,2}\| < 2^{-1}$ for all $j \in \{1 \ldots, k_2\}$, it follows that $\|N_1 - N_2\| < 2^{-1}$ when we view N_1 as an element of $\mathbb{M}_{2^{\ell_2}}(\mathbb{C})$ under the standard embedding.

By repeating this construction, we arrive at a strictly increasing sequence $(\ell_n)_{n\geq 1}$ in $\mathbb{N}$, and a sequence $(N_n)_{n\geq 1}$ in $\mathfrak{B}$ such that for each $n \in \mathbb{N}$, we have

- $N_n \in \mathbb{M}_{2^{\ell_n}}(\mathbb{C})$,
- $\|N_n - N_{n+1}\| < 2^{-n}$, and
- $\sigma(N_n) = X_1 \cup \cdots \cup X_n$.

In particular, the sequence $(N_n)_{n\geq 1}$ is Cauchy. If $N \in \mathfrak{B}$ denotes the limit of the sequence $(N_n)_{n\geq 1}$, then upper semicontinuity quickly implies that $X_n \subseteq \sigma(N)$ for all $n \in \mathbb{N}$, and thus $X \subseteq \sigma(N)$. To see that equality holds, suppose to the contrary that there exists an element $\lambda \in \sigma(N) \setminus X$. Then there exists an open set U containing λ and such that $U \cap X = \emptyset$. In this case, Lemma 2.2.4 ensures the existence of a positive integer n_0 such that $\sigma(N_n) \cap U \neq \emptyset$ whenever $n \geq n_0$. Since $\sigma(N_n) \subseteq X$, we obtain a contradiction and conclude that $\sigma(N) = X$.

■

Before moving to the main result of the section, we mention one final lemma. This fact can be deduced by elementary K-theoretic arguments, and the reader is directed to [4, Lemma 1.2] for a proof.

Lemma 7.2.3. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra. If $V \in \mathfrak{A}$ is an isometry and $U \in \mathcal{U}(\mathfrak{A})$, then*

$$[U]_1 = [VUV^* + (I - VV^*)]_1 .$$

The proof of Theorem 7.2.5 will require a strong version of Theorem 6.3.4 in the case of normal operators with equal (not necessarily trivial) index functions, which follows from the result of Dadarlat [5, Theorem 1.7] described in Section 6. As the level of K-theory this theorem involves is well beyond the scope of this paper, we state Dadarlat's result below and refer the reader to [16] for the required definitions.

Theorem 7.2.4 (Dadarlat)**.** *Let X be a compact metric space, let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, and let $\varphi, \psi : \mathcal{C}(X) \to \mathfrak{A}$ be two unital, injective $*$-homomorphisms. Then φ and ψ are approximately unitarily equivalent if and only if $[[\varphi]] = [[\psi]]$ in $KL(\mathcal{C}(X), \mathfrak{A})$.*

One important consequence of this fact is that two normal operators in a unital, simple, purely infinite C^*-algebra are approximately unitarily equivalent if and only if they have identical spectra, identical index functions, and equivalent common spectral projections. This allows us to prove the following theorem:

Theorem 7.2.5. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N, M \in \mathrm{Nor}(\mathfrak{A})$ be such that*

(1) $\sigma(M) \subseteq \sigma(N)$,

(2) $\Gamma(M)(\lambda) = \Gamma(N)(\lambda)$ *for all* $\lambda \in \mathbb{C} \setminus \sigma(N)$, *and*

(3) *N and M have equivalent common spectral projections.*

Then

$$\mathrm{dist}(\mathcal{U}(N), \mathcal{U}(M)) = d_H(\sigma(N), \sigma(M)).$$

Proof. By Proposition 7.1.2, we need only show that $d_H(\sigma(N_1), \sigma(N_2))$ is an upper bound for the distance between unitary orbits. Let $V \in \mathfrak{A}$ be a non-unitary isometry, and define $P := VV^*$. Let $\mathfrak{B}$ denote the copy of the 2^∞-UHF C^*-algebra inside $\mathfrak{C} := (I-P)\mathfrak{A}(I-P)$, whose existence is guaranteed by Lemma 7.2.1. Further, Lemma 7.2.2 ensures that $\mathfrak{B}$ contains normal operators N_0 and M_0 such that $\sigma(N_0) = \sigma(N)$ and $\sigma(M_0) = \sigma(M)$. Using these operators, we define

$$N' := VMV^* + N_0 \quad \text{and} \quad M' := VMV^* + M_0,$$

and note that with respect to the decomposition $P\mathcal{H} \oplus (I-P)\mathcal{H}$,

$$N' = \begin{bmatrix} VMV^* & 0 \\ 0 & N_0 \end{bmatrix} \quad \text{and} \quad M' = \begin{bmatrix} VMV^* & 0 \\ 0 & M_0 \end{bmatrix}.$$

From this description and by our choice of N_0, it is easy to see that $\sigma(N') = \sigma(N)$. We shall demonstrate that N and N' have equal index functions and equivalent common spectral projections.

Suppose that f is a function on $\mathbb{C}$ that is analytic on an open set U containing $\sigma(N)$, and such that $f(U) \subseteq \{0,1\}$. As f may be approximated by Taylor polynomials, each of which satisfies $p(VMV^*) = Vp(M)V^*$, it follows that $f(VMV^*) = Vf(M)V^*$, and hence

$$f(N') = \begin{bmatrix} f(VMV^*) & 0 \\ 0 & f(N_0) \end{bmatrix} = \begin{bmatrix} Vf(M)V^* & 0 \\ 0 & f(N_0) \end{bmatrix} = Vf(M)V^* + f(N_0). \qquad (*)$$

From here we will consider two separate cases. If $f(M) = 0$, then the fact that M and N were assumed to have equivalent common spectral projections implies that $f(M) \sim_0 f(N)$, and hence $f(N) = 0$. This, in particular, demonstrates that f vanishes on $\sigma(N) = \sigma(N_0)$, and we obtain $f(N') = f(N_0) = 0$, so $f(N') \sim_0 f(N)$. Otherwise, $f(M) \neq 0$, and so $f(N) \neq 0$ by equation $(*)$. Since Lemma 7.2.1 states that every non-zero projection in $\mathfrak{B}$ is trivial in $K_0(\mathfrak{A})$, it is immediate that $[f(N_0)]_0 = 0$ in $\mathfrak{A}$, and we obtain

$$\left[f(N')\right]_0 = [Vf(M)V^*]_0 + [f(N_0)]_0 = [f(M)]_0 = [f(N)]_0\,.$$

Again, we see that $f(N') \sim_0 f(N)$, and conclude that N' and N have equivalent common spectral projections.

As for the index function assumption of Theorem 6.3.4, first notice that as a UHF C^*-algebra, $\mathfrak{B}$ has the property that $\mathrm{GL}(\mathfrak{B})$ is connected. It follows that $N_0 - \lambda I \in \mathrm{GL}(\mathfrak{B})_0$ for every $\lambda \in \mathbb{C} \setminus \sigma(N)$, and hence $N' - \lambda I$ belongs to the same connected component of $\mathrm{GL}(\mathfrak{A})$ as

$$V(M - \lambda I)V^* + (\lambda I - P).$$

However, Lemma 7.2.3 implies that $V(M - \lambda I)V^* + (\lambda I - P)$ belongs to the same connected component of $\mathrm{GL}(\mathfrak{A})$ as $M - \lambda I$. This demonstrates that $\Gamma(N')(\lambda) = \Gamma(M)(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \sigma(N)$. Since it has been assumed that $\Gamma(M)(\lambda) = \Gamma(N)(\lambda)$ for all such λ, we conclude that $\Gamma(N') = \Gamma(N)$.

These facts in tandem with the remarks following Theorem 7.2.4 allow one to conclude that $N' \sim_{au} N$, and a similar argument illustrates that $M' \sim_{au} M$. Since two operators are approximately unitarily equivalent if and only if the distance between their unitary orbits is 0, we can easily deduce from the above that

$$\mathrm{dist}(\mathcal{U}(N), \mathcal{U}(M)) = \mathrm{dist}(\mathcal{U}(N'), \mathcal{U}(M')).$$

That being said, given any unitary $U \in \mathfrak{C}$, it is clear that $P + U$ is unitary in $\mathfrak{A}$, and

$$\mathrm{dist}(\mathcal{U}(N'), \mathcal{U}(M')) \leq \|(P + U)N'(P + U) - M'\| = \|UN_0U^* - M_0\|. \qquad (**)$$

Finally, we note that since N_0 and M_0 belong to the unital inclusion of $\mathfrak{B}$ in $\mathfrak{C}$, and since $\Gamma(N_0)$ and $\Gamma(M_0)$ are trivial when we view N_0 and M_0 as elements of $\mathfrak{B}$ (as $\mathrm{GL}(\mathfrak{B})$ is connected), it follows that $\Gamma(N_0)$ and $\Gamma(M_0)$ are also trivial when N_0 and M_0 are viewed as elements of $\mathfrak{C}$. Moreover, the fact that any two non-zero projections in $\mathfrak{B}$ are Murray–von Neumann equivalent in $\mathfrak{A}$, and hence in $\mathfrak{C}$, implies that the assumptions of Corollary 7.1.5 hold when we consider N_0 and M_0 as elements of $\mathfrak{C}$. From this we see that for every $\varepsilon > 0$, there is a unitary $U \in \mathfrak{C}$ such that

$$\|UN_0U^* - M_0\| \leq \varepsilon + d_H(\sigma(N_0), \sigma(M_0)).$$

However, since $\sigma(N_0) = \sigma(N)$, $\sigma(M_0) = \sigma(M)$, and $\varepsilon > 0$ was arbitrary, we may use equation $(**)$ to conclude that

$$\mathrm{dist}(\mathcal{U}(N), \mathcal{U}(M)) \leq d_H(\sigma(N), \sigma(M)),$$

thereby completing the proof.

■

It is here that we make note of an improvement to Theorem 7.2.5. As it happens, the first hypothesis of this theorem may be removed of obtain the following stronger result:

Theorem 7.2.6. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, and let $N_1, N_2 \in \mathrm{Nor}(\mathfrak{A})$ be such that*

(1) $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ *for all* $\lambda \notin \sigma(N_1) \cup \sigma(N_2)$, *and*

(2) N_1 *and* N_2 *have equivalent common spectral projections.*

Then

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

As we only require the use of Theorem 7.2.5 in the discussion for Section 8 (and not the improved result described above), we will refer the reader to [20] for a proof.

8 Closed Similarity Orbits of Normal Operators

Up to this point, our study of normal operators in unital, simple, purely infinite C^*-algebras can be decomposed into two categories: normal limits of nilpotents, and normal operator unitary orbits. It is here that we attempt to bridge the gap between these topics by examining the closed similarity orbits of normal operators in a unital, simple, purely infinite C^*-algebra. Since Theorem 4.1.8 characterized which normal elements in these algebras belong to the closure of nilpotents, by determining the closed similarity orbits of these normal operators, there is potential to obtain information on some non-normal operators within the closure of nilpotents as well.

Although the closures of these similarity orbits are not yet fully understood, Skoufranis [20] was able to characterize which *normal* elements in a unital, simple, purely infinite C^*-algebra are contained in the closed similarity orbit of a given normal operator. This was achieved via application of the results established in Sections 6 and 7, including Theorem 7.2.4. This section will provide an exposition of Skoufranis' result.

We will explore several lemmas before reaching Skoufranis' characterization in Theorem 8.1.5, beginning with the following technical results on similarity in unital C^*-algebras.

Lemma 8.1.1. *Let $\mathfrak{A}$ be a unital C^*-algebra and let $P \in \mathfrak{A}$ be a non-trivial projection. Suppose that $Z \in (I-P)\mathfrak{A}(I-P)$ and $X \in \mathfrak{A}$ is such that $PX(I-P) = X$. If $\lambda \in \mathbb{C}$ is such that $\lambda(I-P) - Z$ is invertible in $(I-P)\mathfrak{A}(I-P)$, then*

$$\lambda P + X + Z \sim \lambda P + Z.$$

Proof. Since $PX(I-P) = X$, we have that X is strictly upper triangular with respect to the decomposition $P\mathcal{H} \oplus (I-P)\mathcal{H}$. Let $X_1 \in \mathfrak{A}$ denote the upper-right block of X in this decomposition. Define $Y := X(\lambda(I-P) - Z)^{-1}$, and $T := I + Y$. Evidently,

$$Y = \begin{bmatrix} 0 & X_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (\lambda I - Z)^{-1} \end{bmatrix} = \begin{bmatrix} 0 & X_1(\lambda I - Z)^{-1} \\ 0 & 0 \end{bmatrix},$$

and noting that $Y^2 = 0$ implies $T^{-1} = I - Y$. Finally, a routine verification that

$$T(\lambda P + X + Z)T^{-1} = \lambda P + Z$$

completes the proof.

■

Corollary 8.1.2. *Let $\mathfrak{A}$ be a unital C^*-algebra, let $n \in \mathbb{N}$, and $\lambda_1, \ldots, \lambda_n$ be distinct elements of $\mathbb{C}$. Suppose that $\{P_j\}_{j=1}^n$ is a collection of non-trivial projections in $\mathfrak{A}$ that sum to I, and $\{A_{i,j}\}_{i,j=1}^n \subseteq \mathfrak{A}$ is such that*

$$A_{i,j} = \begin{cases} 0, & \text{if } i \geq j \\ P_i A_{i,j} P_j, & \text{if } i < j \end{cases}.$$

Then

$$\sum_{j=1}^n \lambda_j P_j + \sum_{i,j=1}^n A_{i,j} \sim \sum_{j=1}^n \lambda_j P_j.$$

Proof. We shall appeal to Lemma 8.1.1 with

$$P := P_1 \ , \ \ Z := \sum_{j=2}^n \lambda_j P_j + \sum_{i,j=2}^n A_{i,j} \ , \ \text{ and } \ X := \sum_{j=1}^n A_{1,j}.$$

Since the spectrum of Z as an element of $(I-P)\mathfrak{A}(I-P)$ is given by $\{\lambda_2,\dots,\lambda_n\}$, we may define $\lambda := \lambda_1$ and obtain

$$\sum_{j=1}^{n}\lambda_j P_j + \sum_{i,j=1}^{n} A_{i,j} \sim \sum_{j=1}^{n}\lambda_j P_j + \sum_{i,j=2}^{n} A_{i,j} = \lambda_1 P_1 \oplus \left(\sum_{j=2}^{n}\lambda_j P_j + \sum_{i,j=2}^{n} A_{i,j}\right).$$

We now repeat the above analysis in the unital C^*-algebra $(I-P_1)\mathfrak{A}(I-P_1)$ to show that

$$\sum_{j=2}^{n}\lambda_j P_j + \sum_{i,j=2}^{n} A_{i,j} \sim \sum_{j=2}^{n}\lambda_j P_j + \sum_{i,j=3}^{n} A_{i,j} = \lambda_2 P_2 \oplus \left(\sum_{j=3}^{n}\lambda_j P_j + \sum_{i,j=3}^{n} A_{i,j}\right).$$

The result now follows by induction.

■

Using these facts together with the construction of the 2^∞-UHF C^*-algebra from Section 7, we make our first stride in obtaining a characterization of the normal elements lying in the closure of a normal operator similarity orbit by proving the following lemma:

Lemma 8.1.3. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, let $M \in \mathfrak{A}$, and let $V \in \mathfrak{A}$ be a non-unitary isometry. Further, let $P := VV^*$ and $\mathfrak{B}$ be the unital copy of the 2^∞-UHF C^*-algebra in $(I-P)\mathfrak{A}(I-P)$ constructed in Lemma 7.2.1. If μ is a cluster point of $\sigma(M)$ and $Q \in \mathbb{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$ is a nilpotent matrix for some $\ell \in \mathbb{N}$, then $VMV^* + \mu(I-P) + Q \in \overline{\mathcal{S}(M)}$.*

Proof. Since $Q \in \mathbb{M}_{2^\ell}(\mathbb{C})$ is nilpotent, and hence unitarily equivalent to a strictly upper triangular matrix, we may assume that Q is strictly upper triangular. Let $\{\mu_j\}_{j\geq 1}$ be a sequence of distinct elements in $\sigma(M)$ converging to μ. For each $n \in \mathbb{N}$, let

$$T_n := \mathrm{diag}(\mu_n, \mu_{n+1}, \dots, \mu_{n+2^\ell-1}) \in \mathbb{M}_{2^\ell}(\mathbb{C})$$

and note that $T_n \to \mu(I-P)$ as $n \to \infty$. Finally, if we define $M_n := VMV^* + T_n \in \mathfrak{A}$ for each $n \in \mathbb{N}$, then we may verify by arguments identical to those in Theorem 7.2.5 that each M_n is approximately unitarily equivalent to M. However, if for each $i,j \in \{1,\dots,2^\ell\}$ we let $A_{i,j}$ denote the $(i,j)^{th}$ entry of Q, P_i denote the i^{th} orthogonal minimal projection in $\mathbb{M}_{2^\ell}(\mathbb{C})$, and $\lambda_i := \mu_{n+i-1}$, then Corollary 8.1.2 implies that

$$M_n \sim VMV^* + (T_n + Q)$$

for every $n \in \mathbb{N}$. Combining these observations, we see that $M \sim VMV^* + (T_n+Q)$ for every $n \in \mathbb{N}$, and since $T_n + Q \to \mu(I-P)+Q$, we may conclude that $VMV^* + \mu(I-P) + Q \in \overline{\mathcal{S}(M)}$.

■

With Lemma 8.1.3 in hand, one can now deduce the following result which serves as the backbone for the proof of Theorem 8.1.5.

Lemma 8.1.4. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra. Let N and M be normal elements of $\mathfrak{A}$, and write $\sigma(N) = K_1 \sqcup K_2$ where K_1 and K_2 are compact, and K_1 is connected. Suppose that*

(1) *$\sigma(M) = K_1' \cup K_2$ where $K_1' \subseteq K_1$,*

(2) *$\Gamma(N)(\lambda) = \Gamma(M)(\lambda)$ for all $\lambda \in \mathbb{C}\setminus\sigma(N)$, and*

(3) *N and M have equivalent common spectral projections.*

Then $N \in \overline{\mathcal{S}(M)}$ whenever K_1' contains a cluster point of $\sigma(M)$.

Proof. First, consider the case that K_1 is a singleton. It it immediate that $K_1' = K_1$ and hence $\sigma(N) = \sigma(M)$. An application of Theorem 7.2.5 now demonstrates that

$$d(\mathcal{U}(N), \mathcal{U}(M)) = d_H(\sigma(N), \sigma(M)) = 0,$$

and it follows that $N \sim_{au} M$.

Next, suppose that K_1' is not a singleton. Let $\mu \in K_1'$ be a cluster point of $\sigma(M)$ and $\varepsilon > 0$. Let $V \in \mathfrak{A}$ be a non-unitary isometry, define $P := VV^*$, and let $\mathfrak{B}$ denote the copy of the 2^∞-UHF C^*-algebra inside $(I-P)\mathfrak{A}(I-P)$ constructed in Lemma 7.2.1. Using Theorem 5.2.1, we may obtain a normal operator $T \in \mathfrak{B}$ that is a norm limit of nilpotent operators in $\mathfrak{B}$ (and hence a norm limit of nilpotent matrices in $\bigcup_{\ell \geq 1} \mathbb{M}_{2^\ell}(\mathbb{C})$ by Proposition 5.1.10), and such that $\sigma(T) = \{z \in \mathbb{C} : |z| \leq \varepsilon\}$. Note that if Q is any nilpotent matrix in $\bigcup_{\ell \geq 1} \mathbb{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$, then

$$VMV^* + \mu(I-P) + Q \in \overline{\mathcal{S}(M)}$$

by Lemma 8.1.3. Consequently, the normal operator

$$M_1 := VMV^* + \mu(I-P) + T$$

belongs to $\overline{\mathcal{S}(M)}$, as T is a norm limit of nilpotent matrices from $\bigcup_{\ell \geq 1} \mathbb{M}_{2^\ell}(\mathbb{C})$.

We may now argue as in the proof of Theorem 7.2.5 to conclude that M_1 and N have equivalent common spectral projections, and

$$\Gamma(M_1)(\lambda) = \Gamma(M)(\lambda) = \Gamma(N)(\lambda)$$

for all $\lambda \notin \sigma(M_1) \cup \sigma(N)$. Since $\sigma(M_1) = \sigma(M) \cup \{z \in \mathbb{C} : |z-\mu| \leq \varepsilon\}$ and $\mu \in K_1$, the connectedness of K_1 together with the fact that K_1 is not a singleton suggests that $K_1 \cap \{z \in \mathbb{C} : |z-\mu| \leq \varepsilon\}$ contains a cluster point $\mu' \neq \mu$ of $\sigma(M_1)$. The above arguments may now be repeated with M replaced by M_1 to obtain a normal operator $M_2 \in \overline{\mathcal{S}(M_1)} \subseteq \overline{\mathcal{S}(M)}$ such that

$$\sigma(M_2) = \sigma(M_1) \cup \{z \in \mathbb{C} : |z-\mu'| \leq \varepsilon\},$$

$\Gamma(M_2)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(M_2) \cup \sigma(N)$, and M_2 and N have equivalent common spectral projections.

After a finite number of repetitions, we arrive at a normal operator $M_0 \in \overline{\mathcal{S}(M)}$ that satisfies

- $\sigma(M_0) = K_1'' \cup K_2$ where K_1'' is connected and $K_1 \subseteq K_1'' \subseteq \{z \in \mathbb{C} : \mathrm{dist}(z, K_1) \leq \varepsilon\}$,
- M_0 and N have equivalent common spectral projections, and
- $\Gamma(M_0)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(M_0)$.

We now appeal to Theorem 7.2.5 to conclude that

$$\mathrm{dist}(\mathcal{U}(N), \mathcal{U}(M_0)) = d_H(\sigma(N), \sigma(M_0)) \leq \varepsilon.$$

Therefore $\mathrm{dist}(N, \mathcal{S}(M)) \leq \varepsilon$, and we obtain that $N \in \overline{\mathcal{S}(M)}$, as claimed. ∎

Theorem 8.1.5. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N, M \in \mathrm{Nor}(\mathfrak{A})$. Then $N \in \overline{\mathcal{S}(M)}$ if and only if*

(1) $\sigma(M) \subseteq \sigma(N)$,

(2) $\sigma(M)$ *intersects every connected component of* $\sigma(N)$,

(3) $\Gamma(N)(\lambda) = \Gamma(M)(\lambda)$ *for all* $\lambda \in \mathbb{C} \setminus \sigma(N)$,

(4) *if* $\lambda \in \sigma(N)$ *is not an isolated point, then the connected component of* $\sigma(N)$ *that contains* λ *also contains some non-isolated point of* $\sigma(M)$, *and*

(5) N *and* M *have equivalent common spectral projections.*

Proof. For the "only if" direction, notice that the necessity of conditions (1) and (2) is immediate from Proposition 6.1.2, and that (3) is necessary is a result of Lemma 6.1.7. To see that (5) is required, let f be a function that is analytic on an open set U containing $\sigma(N)$ and such that $f(U) \subseteq \{0, 1\}$. Then Lemma 6.3.2 guarantees that $f(N) \in \overline{\mathcal{S}(f(M))}$, and Lemma 6.3.1 can now be used to argue that $f(N) \sim_0 f(M)$.

For the necessity of (4), let K_λ denote the connected component of $\sigma(N)$ that contains λ. Note first that if K_λ is not isolated in $\sigma(N)$, then every open neighbourhood of K_λ must intersect a different connected component of $\sigma(N)$. Since each of these components must in turn contain a point of $\sigma(M)$ by hypothesis (2), $\sigma(M) \cap K_\lambda$ must contain a cluster point of $\sigma(M)$. This point is clearly non-isolated in $\sigma(M)$. On the other hand, suppose that K_λ is isolated in $\sigma(N)$. To obtain a contradiction, suppose that $\sigma(M) \cap K_\lambda$ does not contain a cluster point of $\sigma(M)$. Define ι_{K_λ} to be the identity function on K_λ, and 0 on $\sigma(N) \setminus K_\lambda$. We may extend ι_{K_λ} to an analytic function on an open neighbourhood of $\sigma(N)$, and hence by appealing to Lemma 6.3.2, it is immediate that

$$\iota_{K_\lambda}(N) \in \overline{\mathcal{S}(\iota_{K_\lambda}(M))}.$$

However, the fact that $\sigma(M) \cap K_\lambda$ does not contain a cluster point of $\sigma(M)$ implies that $\sigma(M) \cap K_\lambda$ is a finite set, and hence $\iota_{K_\lambda}(M)$ must have finite spectrum. Let p be a polynomial that vanishes on $\sigma(\iota_{K_\lambda}(M))$. If $T \in \overline{\mathcal{S}(\iota_{K_\lambda}(M))}$, then of course we must have $p(T) = 0$, and hence it follows from the above analysis that $p(\iota_{K_\lambda}(N)) = 0$. As K_λ is a connected subset of $\sigma(N)$ that is not a singleton, this implies that p has infinitely many roots, which gives the desired contradiction.

Suppose now that conditions (1) – (5) hold and let $\varepsilon > 0$. Let $L_1, \ldots, L_n$ be finitely many connected components of $\sigma(N)$ so that

$$\sigma(N) \subseteq \bigcup_{i=1}^{n} \{z \in \mathbb{C} : \mathrm{dist}(z, L_i) \leq \varepsilon\}.$$

For each $i \in \{1, \ldots, n\}$, let χ_{L_i} denote the characteristic function on L_i. By Lemma 8.1.4, it is clear that

$$M_1 := M - \chi_{L_1}(M)M + \chi_{L_1}(N)N$$

belongs to $\overline{\mathcal{S}(M)}$. Similarly, we may deduce that

$$M_2 := M_1 - \chi_{L_2}(M_1)M_1 + \chi_{L_2}(N)N$$

belongs to $\overline{\mathcal{S}(M_1)} \subseteq \overline{\mathcal{S}(M)}$. Applying this process a finite number of times leads to a normal operator $M' \in \overline{\mathcal{S}(M)}$ with $d_H(\sigma(N), \sigma(M')) \leq \varepsilon$, and such that N and M' satisfy hypotheses (1) – (5) of this theorem. It now follows from Theorem 7.2.5 that

$$\mathrm{dist}(\mathcal{U}(N), \mathcal{U}(M')) = d_H(\sigma(N), \sigma(M')) \leq \varepsilon.$$

This implies that $\text{dist}(N, \mathcal{S}(M)) \leq \varepsilon$, thereby completing the proof.

■

The following corollary is an immediate consequence of the above result together with Theorem 7.2.4 and its proceeding remarks.

Corollary 8.1.6. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \text{Nor}(\mathfrak{A})$. If $N_1 \in \overline{\mathcal{S}(N_2)}$ and $N_2 \in \overline{\mathcal{S}(N_1)}$, then $N_1 \sim_{au} N_2$.*

Proof. Theorem 8.1.5 can be applied to conclude that $\sigma(N_1) = \sigma(N_2)$, that $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \sigma(N_1)$, and that N_1 and N_2 have equivalent common spectral projections. Turning to the remarks following Theorem 7.2.4, one quickly observes that $N_1 \sim_{au} N_2$, as desired.

■

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