Assignment #2

AMATH/PMATH 331

Due on 2 February 2011

1. Show the sequence $\{\sqrt[n]{2^n + 3^n}\}$ converges and find its limit.
2. Show the sequence $\{x_n\}$ defined by $a_1 = 0;\ \ a_{n+1} = \sqrt{5 + 2a_n}$ converges and find its limit.
3. If a sequence $\{x_n\}$ satisfies that $x_{n+1} - x_n \to 0$, must $\{x_n\}$ be a Cauchy sequence and thus converge? Justify your answer.
4. Show the sequence $\{x_n\}$ defined recursively by

$$x_1 = 1,\ \ x_{n+1} = \sin x_n,\ \ n = 1, 2, 3, ...$$

   converges, and then find its limit.
5. Determine by the Comparison Test if the following series converge.
   (a) $\sum_{n=1}^{\infty} \frac{\cos n^2}{3^n}$ (b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$
6. (a) Show that the set $\{(x, y) \in \mathbb{R}^2 \mid xy \le 1\}$ is a closed subset of $\mathbb{R}^2$.
   (b) Show that the"closed" ball around a point $\mathbf{a}$ in $\mathbb{R}^n$ of radius $r > 0$

$$\overline{B_r(\mathbf{a})} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| \le r\}$$

   is closed.
7. (a) Show $|\|\mathbf{x}\| - \|\mathbf{y}\|| \le \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
   (b) Show that if $\{\mathbf{x}_k\}$ is a sequence in $\mathbb{R}^n$ such that $\lim \mathbf{x}_k = \mathbf{b}$ then $\lim \|\mathbf{x}_k\| = \|\mathbf{b}\|$.
8. Show that a subset of $\mathbb{R}^n$ is complete if and only if it is a closed subset of $\mathbb{R}^n$.
9. Find the closure of the following sets (do not justify)
   (a) $\mathbb{Q}$
   (b) $\mathbb{R} \backslash \mathbb{Q}$
   (c) $\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q}, y \in \mathbb{R} \backslash \mathbb{Q}\}$
   (d) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + 2y^2 < 1\}$
   (e) $\{(x, y) \in \mathbb{Q}^2 \mid x^2 + 2y^2 < 1\}$
   (f) $\{(x, y) \in \mathbb{R}^2 \mid xy \le 1\}$

10. Is each of the following subset of $\mathbb{R}$ closed, open, both, or neither? If a given set is both open and closed, you must say so. (do not justify)

   (a) $S = \bigcup_{k=1}^{\infty}(-k, k)$
   (b) $S = \bigcup_{k=1}^{\infty}(\frac{1}{k}, 3 - \frac{1}{k})$
   (c) $S = \bigcup_{k=1}^{\infty}(\frac{1}{k}, 3 - \frac{1}{k}]$
   (d) $S = \bigcup_{k=1}^{\infty}[\frac{1}{k}, 3 - \frac{1}{k}]$
   (e) $S = \bigcap_{k=1}^{\infty}(-k, k)$
   (f) $S = \bigcap_{k=1}^{\infty}(-\frac{1}{k}, 3 + \frac{1}{k})$
   (g) $S = \bigcap_{k=1}^{\infty}(-\frac{1}{k}, 3 + \frac{1}{k}]$
   (h) $S = \bigcap_{k=1}^{\infty}[\frac{1}{k}, 3 + \frac{1}{k})$

11. Prove that $\mathbb{Q}$ is neither closed nor open.

12. For each set in the following, say whether it is compact or not. Give a BRIEF explanation.

   (a) $A = (0, 1]$ in $\mathbb{R}$.
   (b) $A = [0, 1] \times [0, 1)$ in $\mathbb{R}^2$.
   (c) $A = [a, b] \times [c, d]$ in $\mathbb{R}^2$ where $a < b$ and $c < d$.
   (d) $A = \{(x, y) \in \mathbb{R}^2 \mid xy \leq 1\}$.
   (e) $A = \bigcup_{n=1}^{\infty} I_n$ where $I_n = [\frac{1}{n+1}, 1]$ in $\mathbb{R}$.
   (f) $A = \bigcup_{n=1}^{\infty} I_n$ where $I_n = [-1, \frac{1}{n+1})$ in $\mathbb{R}$.
   (g) $\bigcap_{n=1}^{\infty} F_n$ where $F_n = \{x \in \mathbb{R} \mid x > 0, 1 \leq x^2 \leq 2 + \frac{1}{n}\}$.
   (h) $A = \mathbb{Q} \bigcap [0, 1]$ in $\mathbb{R}$.
   (i) $A = \{(x, y) \in \mathbb{R}^2 \mid y = \sin x\}$.
   (j) $A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$.
   (k) $A = [0, 1] \backslash C$ where C is the Cantor set.
   (l) $A = \{0\} \bigcup \{\frac{1}{k} \mid k \in \mathbb{N}\}$ in $\mathbb{R}$.
   (m) $A = \mathbb{Z}$.

13. Give an example to show that Cantor's Intersection Theorem would not be true if compact sets were replaced by closed sets.

14. Show that the union of finitely many compact sets is compact.

15. Give a counter-example to the following statement: the union of an arbitrary family of compact sets is compact.

16. Let A and B be *disjoint* closed subsets of $\mathbb{R}^n$ (their being disjoint means $A \cap B = \emptyset$). Define

$$d(A, B) = \inf\{\|\mathbf{a} - \mathbf{b}\| \mid \mathbf{a} \in A, \mathbf{b} \in B\}.$$

   (a) If $A = \{\mathbf{a}\}$ is a singleton, show that $d(A, B) > 0$.
   (b) If A is compact, show that $d(A, B) > 0$.