Solutions to the Problems on Sequences, Series and Products

1: Let $a_1 = 1$ and for $n \geq 1$ let $a_{n+1} = \dfrac{6}{a_n + 1}$. Determine whether $\{a_n\}$ converges, and if so then find the limit.

Solution: Note that $a_{n+2} = \dfrac{6}{a_{n+1}+1} = \dfrac{6}{\frac{6}{a_n+1}+1} = \dfrac{6a_n+6}{a_n+7} = 6 - \dfrac{36}{a_n+7}$. If the sequence of odd terms $\{a_{2k+1}\}$ converges with $a_{2k+1} \to l$, then we also have $a_{2k+3} \to l$, so by taking the limit on both sides of the recurrence equation $a_{2k+3} = 6 - \dfrac{36}{a_{2k+1}+7}$ we have $l = 6 - \dfrac{36}{l+7} \implies (l-6)(l+7)+36 = 0 \implies l^2 + l - 6 = 0 \implies l = -3$ or $l = 2$. Note that $a_1 = 1$ and $a_3 = 6 - \frac{36}{1+7} = \frac{3}{2}$. We claim that for all odd values of $n \geq 1$, we have $a_n < a_{n+2} < 2$. The base case holds, so suppose inductively that $n \geq 1$ is odd and $a_n < a_{n+2} < 2$. Then we have $a_n < a_{n+2} < 2 \implies a_n + 7 < a_{n+2} + 7 < 9 \implies \frac{1}{a_n+7} > \frac{1}{a_{n+2}+7} > \frac{1}{9} \implies -\frac{36}{a_n+7} < -\frac{36}{a_{n+2}+7} < -4 \implies 6 - \frac{36}{a_n+7} < 6 - \frac{36}{a_{n+2}+7} < 2 \implies a_{n+2} < a_{n+4} < 2$. Thus the claim holds, so the sequence $\{a_{2k+1}\}$ is increasing and bounded above by 2, so it converges to some limit l. We saw above that $l = -3$ or $l = 2$, and since $a_1 = 1$ and the sequence $\{a_{2k+1}\}$ is increasing, the limit must be $l = 2$. A similar argument shows that the sequence of even terms $\{a_{2k}\}$ is decreasing with limit $l = 2$, and it follows that $\{a_n\}$ converges with limit $l = 2$.

2: Let $a_1 = 1$, $a_2 = 2$, and for $n > 2$ let $a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$. Determine whether $\{a_n\}$ converges, and if so then find the limit.

Solution: Note that if $\{a_n\}$ does converge with $a_n \to l$, then we also have $a_{n-1} \to l$ and $a_{n-2} \to l$, and so taking the limit on both sides of of the recursion formula $a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$ gives $l = 2\sqrt{l}$, so $l^2 = 4l$ and so $l = 0$ or $l = 4$. Also note that, by induction, $a_n \geq 1$ for all n. Let $b_n = |4 - a_n|$. Then $b_n = \left|4 - \sqrt{a_{n-1}} - \sqrt{a_{n-2}}\right| = \left|(2 - \sqrt{a_{n-1}}) + (2 - \sqrt{a_{n-2}})\right| = \left|\dfrac{4 - a_{n-1}}{2 + \sqrt{a_{n-1}}} + \dfrac{4 - a_{n-2}}{2 + \sqrt{a_{n-2}}}\right| \leq \dfrac{|4 - a_{n-1}|}{2 + \sqrt{a_{n-1}}} + \dfrac{|4 - a_{n-2}|}{2 + \sqrt{a_{n-2}}} \leq \dfrac{b_{n-1}}{3} + \dfrac{b_{n-2}}{3}$. If we had $b_n = \frac{1}{3}b_{n-1} + \frac{1}{3}b_{n-2}$, that is $3b_n - b_{n-1} - b_{n-2}$, then by solving $3\lambda^2 - \lambda - 1 = 0$ to get $\lambda = \frac{1 \pm \sqrt{13}}{6}$, we would obtain $b_n = a\left(\frac{1+\sqrt{13}}{6}\right)^n + b\left(\frac{1-\sqrt{13}}{6}\right)^n$ for some constants a and b. Since we actually have $b_n \leq \frac{1}{3}b_{n-1} + \frac{1}{3}b_{n-2}$, we obtain $b_n \leq a\left(\frac{1+\sqrt{13}}{6}\right)^n + b\left(\frac{1-\sqrt{13}}{6}\right)^n$, and so $b_n \to 0$ as $n \to \infty$, and hence $a_n \to 4$ as $n \to \infty$.

3: Let $a_1 = \sqrt{2}$, for $n \geq 1$ let $a_{n+1} = \sqrt{2 + a_n}$, and then let $b_n = 4^n(2 - a_n)$. Determine whether $\{b_n\}$ converges, and if so then find the limit.

Solution: Note that by repeatedly applying the identity $\cos\frac{\theta}{2} = \frac{\sqrt{2+2\cos\theta}}{2}$ we obtain $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$, $\cos\frac{\pi}{16} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$ and so on, so $a_n = 2\cos\frac{\pi}{2^{n+1}}$. Thus $b_n = 4^n(2 - a_n) = 4^n\left(2 - 2\cos\frac{\pi}{2^{n+1}}\right) = 4^n\left(4\sin^2\frac{\pi}{2^{n+2}}\right) = \frac{\pi^2}{4}\left(\frac{\sin\frac{\pi}{2^{n+2}}}{\frac{\pi}{2^{n+2}}}\right)^2 \to \frac{\pi^2}{4}$ as $n \to \infty$.

4: Let $a_n = \left(\dfrac{n^n}{n!}\right)^{1/n}$. Determine whether $\{a_n\}$ converges, and if so then find the limit.

Solution: By the Root Test, $\lim\limits_{n\to\infty} a_n$ is equal to the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} \dfrac{n!}{n^n}x^n$. By the Ratio Test, this radius of convergence is also equal to $\lim\limits_{n\to\infty} \dfrac{(n+1)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^n} = \lim\limits_{n\to\infty}\left(\dfrac{n+1}{n}\right)^n = e$.

5: For a real number x, let $\langle x\rangle$ denote the fractional part of x, that is $\langle x\rangle = x - \lfloor x\rfloor$. Show that if x is irrational then the sequence $\big\{ \langle nx\rangle \big\}$ is dense in $[0,1]$.

Solution: Let x be irrational. We claim that given any $\epsilon > 0$, we can find a positive integer m such that $\langle mx\rangle \in (0,\epsilon)\cup(1-\epsilon,1)$. Let $\epsilon > 0$. Choose an integer $p > \frac{1}{\epsilon}$, then divide $[0,1]$ into p equal-sized subintervals. By the Pigeonhole Principle we can choose k and l with $k < l$ so that $\langle kx\rangle$ and $\langle lx\rangle$ both lie in the same subinterval. Then we have $\big|\langle kx\rangle - \langle lx\rangle\big| \leq \frac{1}{p} < \epsilon$. Note that if we then set $m = l - k$ we either have $\langle mx\rangle < \epsilon$ or we have $\langle mx\rangle > 1-\epsilon$. Since x is irrational, mx is not an integer so $\langle mx\rangle \neq 0$, and so we have $\langle mx\rangle \in (0,\epsilon)\cup(1-\epsilon,1)$, as claimed.

To show that $\big\{ \langle nx\rangle \big\}$ is dense, we must show that given a point $a\in[0,1]$ and given $\epsilon > 0$ it is possible to find a value of n such that $|\langle nx\rangle - a| < \epsilon$. Let $a \in [0,1]$ and let $\epsilon > 0$. Choose m so that $\langle mx\rangle \in (0,\epsilon)\cup(1-\epsilon,1)$. If $m \in (0,\epsilon)$, then let $r = \langle mx\rangle < \epsilon$, and notice that for $1 \leq k \leq \lfloor \frac{1}{r}\rfloor$ we have $\langle kmx\rangle = k\langle mx\rangle = kr$, and that one of the numbers $r, 2r, 3r, \cdots, \lfloor\frac{1}{r}\rfloor r$ will be within a distance of ϵ from a. If $m \in (1-\epsilon, 1)$, then let $r = 1 - \langle mx\rangle < \epsilon$, and notice that for $1 \leq k \leq \lfloor\frac{1}{r}\rfloor$ we have $\langle kmx\rangle = 1 - k(1-\langle mx\rangle) = 1 - kr$ and that one of the numbers $1-r, 1-2r, 1-3r, \cdots, 1-\lfloor\frac{1}{r}\rfloor r$ will be within a distance of ϵ from a. In either case, we can choose k so that $\big|\langle kmx\rangle - a\big| < \epsilon$.

6: (a) Find $\displaystyle\sum_{k=2}^{n} \frac{1}{\log_k e}$.

Solution: Note that $\log_k e = \dfrac{\ln e}{\ln k} = \dfrac{1}{\ln k}$, so $\displaystyle\sum_{k=2}^{n}\frac{1}{\log_k e} = \sum_{k=2}^{n}\ln k = \ln\left(\prod_{k=2}^{n} k\right) = \ln(n!)$.

(b) Find $\displaystyle\sum_{k=1}^{n}(2k-1)^3$.

Solution: Recall that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{2}$, so we have $\sum_{k=1}^{n}(2k-1)^3 = \sum_{k=1}^{2n} k^3 - \sum_{k=1}^{n}(2k)^3 = \sum_{k=1}^{2n}k^3 - 2^3\sum_{k=1}^{n}k^3 = \frac{(2n)^2(2n+1)^2}{4} - \frac{2^3n^2(n+1)^2}{4} = n^2(2n+1)^2 - 2\,n^2(n+1)^2 = n^2\big((4n^2+4n+1) - 2(n^2+2n+1)\big) = n^2(2n^2-1)$.

7: (a) Find $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i}$.

Solution: $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}\cdot\frac{1}{n} = \int_0^1\frac{dx}{1+x} = \Big[\ln(1+x)\Big]_0^1 = \ln 2.$

(b) Find $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}}$.

Solution: $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{1+\left(\frac{i}{n}\right)^2}}\cdot\frac{1}{n} = \int_0^1\frac{dx}{\sqrt{1+x^2}} = \int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sec\theta} = \int_0^{\pi/4}\sec\theta\,d\theta = \Big[\ln(\sec\theta+\tan\theta)\Big]_0^{\pi/4} = \ln(\sqrt{2}+1)$, where we made the change of variables $\tan\theta = x$.

8: (a) Find $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3n-2}$.

Solution: For $|x|<1$ we have $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{3n}=\frac{1}{1+x^3}$, so $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{3n+1}=\int_0^x\frac{dt}{1+t^3}$. By Abel's Theorem we can put in $x=1$ to get $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}=\int_0^1\frac{dt}{1+t^3}$. Thus $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3n-2}=\tfrac12+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n-2}=\tfrac12+\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}=\tfrac12+\int_0^1\frac{dt}{t^3+1}$. We can solve this integral using partial fractions. To get $\displaystyle\frac{1}{t^3+1}=\frac{A}{t+1}+\frac{B(2t-1)+C}{t^2-t+1}$, we need $A(t^2-t+1)+B(2t^2+t-1)+C(t+1)=1$. Equate coefficients to get the three equations $A+2B=0$, $-A+B+C=0$ and $A-B+C=1$. Solve these to get $A=\frac13$, $B=-\frac16$ and $C=\frac12$. Thus we find that $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3n-2}=\tfrac12+\int_0^1\frac{dt}{t^3+1}=\tfrac12+\int_0^1\frac{\frac13}{t+1}-\frac{\frac16(2t-1)+\frac12}{t^2-t+1}\,dt=\tfrac12+\left[\tfrac13\ln(t+1)-\tfrac16\ln(t^2-t+1)+\tfrac{1}{\sqrt3}\tan^{-1}\tfrac{(t-\frac12)}{\frac{\sqrt3}{2}}\right]_0^1$ $=\frac12+\frac13\ln 2+\frac{1}{\sqrt3}\tan^{-1}\frac{1}{\sqrt3}-\frac{1}{\sqrt3}\tan^{-1}\frac{1}{\sqrt3}=\frac12+\frac13\ln2+\frac{\pi}{3\sqrt3}$.

(b) Find $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^2}{n!}$.

Solution: For all x we have $\displaystyle e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $\displaystyle x\,e^x=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$. Differentiate to get $\displaystyle(x+1)\,e^x=\sum_{n=0}^{\infty}\frac{(n+1)x^n}{n!}$, so $\displaystyle(x^2+x)\,e^x=\sum_{n=0}^{\infty}\frac{(n+1)x^{n+1}}{n!}$. Differentiate again to get $\displaystyle(x^2+3x+1)\,e^x=\sum_{n=0}^{\infty}\frac{(n+1)^2x^n}{n!}$. Put in $x=1$ to get $\displaystyle 5\,e=\sum_{n=0}^{\infty}\frac{(n+1)^2}{n!}=1+\sum_{n=1}^{\infty}\frac{(n+1)^2}{n!}$. Thus $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^2}{n!}=5\,e-1$.

9: Find $\displaystyle\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}k^2\right)^{-1}$.

Solution: Recall that $\displaystyle\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$, so we have $\displaystyle\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}k^2\right)^{-1}=\sum_{n=1}^{\infty}\frac{6}{n(n+1)(2n+1)}$. To get $\displaystyle\frac{A}{n}+\frac{B}{n+1}+\frac{C}{2n+1}=\frac{6}{n(n+1)(2n+1)}$ we need $A(2n^2+3n+1)+B(2n^2+n)+C(n^2+1)=6$. Equate coefficients to get the three equations $2A+2B+C=0$, $3A+B+C=0$ and $A=6$. Solve these to get $A=6$, $B=6$ and $C=-24$. Thus $\displaystyle\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}k^2\right)^{-1}=\sum_{n=1}^{\infty}\frac{6}{n}+\frac{6}{n+1}-\frac{24}{2n+1}$. When n is even, the n^{th} partial sum is $S_n=18-12\left(1-\frac12+\frac13-\frac14+\cdots-\frac1n\right)+\frac{6}{n+1}-24\left(\frac{1}{n+1}+\frac{1}{n+3}+\frac{1}{n+5}+\cdots+\frac{1}{2n-1}\right)-\frac{24}{2n+1}$. To find the limit of the sum $\left(1-\frac12+\frac13-\frac14+\cdots-\frac1n\right)$, note that for $|x|<1$ we have $\frac{1}{1+x}=1-x+x^2-x^3+\cdots$, so $\ln(1+x)=x-\frac12x^2+\frac13x^3-\frac14x^4+\cdots$. By Abel's Theorem, we can put in $x=1$ to get $\ln2=1-\frac12+\frac13-\frac14+\cdots$. The other sum is a Riemann sum: $\left(\frac{1}{n+1}+\frac{1}{n+3}+\frac{1}{n+5}+\cdots+\frac{1}{2n-1}\right)=\frac12\cdot\frac2n\left(\frac{1}{1+\frac1n}+\frac{1}{1+\frac3n}+\cdots+\frac{1}{1+\frac{n-1}{n}}\right)\to$ $\displaystyle\frac12\int_0^1\frac{1}{1+x}\,dx=\frac12\Big[\ln(1+x)\Big]_0^1=\frac12\ln2$. Thus the limit of the even partial sums is $18-24\ln2$. A similar calculation shows that the limit of the odd partial sums is also equal to $18-24\ln2$, so $\displaystyle\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}k^2\right)^{-1}=18-24\ln2$.

10: Find $\int_0^\pi \sin x\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: We have $\int_0^\pi \sin x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{n}\sin\left(\frac{k\pi}{n}\right)$. To find $\sum_{k=1}^{n}\sin\left(\frac{k\pi}{n}\right)$, let $\alpha = e^{i\pi/n}$ so $\sin\frac{k\pi}{n} = \mathrm{Im}(\alpha^k)$. Then $\sum_{k=1}^{n}\sin\frac{k\pi}{n} = \mathrm{Im}\left(\sum_{k=1}^{n}\alpha^k\right) = \mathrm{Im}\left(\frac{\alpha-\alpha^{n+1}}{1-\alpha}\right) = \mathrm{Im}\left(\frac{\alpha(1-\alpha^n)(1-\overline{\alpha})}{1-2\mathrm{Re}(\alpha)+\alpha\overline{\alpha}}\right) = \mathrm{Im}\left(\frac{2(\alpha-\alpha\overline{\alpha})}{1-2\mathrm{Re}(\alpha)+\alpha\overline{\alpha}}\right) = \mathrm{Im}\left(\frac{\alpha-1}{1-\mathrm{Re}(\alpha)}\right) = \frac{\mathrm{Im}(\alpha)}{1-\mathrm{Re}(\alpha)} = \frac{\sin\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}$, since $\alpha^n=-1$ and $\alpha\overline{\alpha}=1$. So $\int_0^\pi \sin x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{n}\sin\left(\frac{k\pi}{n}\right)$ $= \lim_{n\to\infty}\frac{\frac{\pi}{n}\sin\frac{\pi}{n}}{1-\cos\frac{\pi}{n}} = \lim_{x\to 0}\frac{x\sin x}{1-\cos x} = 2$, by using l'Hôpital's Rule twice or by using power series.

11: Let $a_n>0$. Show that $\sum_{n=1}^{\infty}a_n$ converges if and only if $\prod_{n=1}^{\infty}(1+a_n)$ converges.

Solution: If $\sum_{n=1}^{\infty}a_n$ converges then $a_n\to 0$ so $\lim_{n\to\infty}\frac{a_n}{\ln(1+a_n)} = \lim_{x\to 0}\frac{x}{\ln(1+x)} = 1$. If $\sum_{n=1}^{\infty}\ln(1+a_n)$ converges then $\ln(1+a_n)\to 0$ so $(1+a_n)\to 1$ and so $a_n\to 0$ and we again have $\lim_{n\to\infty}\frac{a_n}{\ln(1+a_n)} = 1$. By the Limit Comparison Test, $\sum_{n=1}^{\infty}a_n$ converges if and only if $\sum_{n=1}^{\infty}\ln(1+a_n)$ converges. Also, if we write $P_n = \prod_{k=1}^{n}(1+a_n)$ and $S_n = \sum_{k=1}^{n}\ln(1+a_n)$ then we have $\ln(P_n)=S_n$, so $\{P_n\}$ converges if and only if $\{S_n\}$ converges, that is $\sum_{n=1}^{\infty}\ln(1+a_n)$ converges if and only if $\prod_{n=1}^{\infty}(1+a_n)$ converges.

12: Find $\prod_{n=0}^{\infty}\left(1+\frac{1}{2^{2^n}}\right)$.

Solution: Let $P_n = \prod_{k=0}^{n}\left(1+\frac{1}{2^{2^k}}\right)$. Then $\left(1-\frac12\right)P_n = \left(1-\frac12\right)\left(1+\frac12\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^4}\right)\cdots\left(1+\frac{1}{2^{2^n}}\right) = \left(1-\frac{1}{2^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^4}\right)\cdots\left(1+\frac{1}{2^{2^n}}\right) = \left(1-\frac{1}{2^4}\right)\left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{2^8}\right)\cdots\left(1+\frac{1}{2^{2^n}}\right) = \cdots = \left(1-\frac{1}{2^{2^n}}\right)\left(1+\frac{1}{2^{2^n}}\right)$ $= \left(1-\frac{1}{2^{2^{n+1}}}\right)$. Thus $P_n = \frac{1-\frac{1}{2^{2^{n+1}}}}{1-\frac12} \to \frac{1}{1-\frac12} = 2$ as $n\to\infty$.

13: Find $\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$.

Solution: Let $P_n = \prod_{k=2}^{n}\frac{k^3-1}{k^3+1}$. Then $P_n = \prod_{k=2}^{n}\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)} = \prod_{k=2}^{n}\frac{k-1}{k+1}\prod_{k=2}^{n}\frac{k^2+k+1}{(k-1)^2+(k-1)+1} =$ $\left(\frac13\cdot\frac24\cdot\frac35\cdots\frac{k-2}{k}\cdot\frac{k-1}{k+1}\right)\left(\frac73\cdot\frac{13}{7}\cdot\frac{21}{13}\cdots\frac{(k-1)^2+(k-1)+1}{(k-2)^2+(k-2)+1}\cdot\frac{k^2+k+1}{(k-1)^2+(k-1)+1}\right) = \left(\frac{1\cdot 2}{k(k+1)}\right)\left(\frac{k^2+k+1}{3}\right) \to \frac23$ as $n\to\infty$.