Lesson 5: Polynomials

1: Let $p(x)$ be a polynomial over $\mathbf{Z}$ with at least four distinct integral roots. Show that there is no integer k such that $p(k)$ is prime.

2: Let $p(x)$ be a polynomial over $\mathbf{C}$. Show that $p(x)$ is even if and only if there exists a polynomial $q(x)$ over $\mathbf{C}$ such that $p(x) = q(x)q(-x)$.

3: Let $p(x)$ be a polynomial over $\mathbf{R}$ of odd degree. Show that $p(p(x))$ has at least as many real roots as $p(x)$.

4: Let $p(x)$ be a monic polynomial over $\mathbf{Z}$ with the property that there exist positive integers k and l such that none of the integers $p(k+i)$ with $i = 1, 2, \cdots, l$ is divisible by l. Show that $p(x)$ has no rational roots.

5: Let $p(x) = \sum_{k=0}^{2n} (-1)^k (2n+1-k)\, x^k$. Show that $p(x)$ has no real roots.

6: Let $p(x)$ be a polynomial with non-negative real coefficients. Show that $p(a^2)p(b^2) \geq p(ab)^2$ for all $a, b \in \mathbf{R}$.

7: Let $p(x)$ be a polynomial over $\mathbf{Z}$ of degree at least 2. Show that there is a polynomial $q(x)$ over $\mathbf{Z}$ such that $p(q(x))$ is reducible over $\mathbf{Z}$.

8: Let a and b be distinct real numbers. Solve $(z-a)^4 + (z-b)^4 = (a-b)^4$ for $z \in \mathbf{C}$.

9: Let $a_1, a_2, \cdots, a_n$ be distinct integers. Show that $p(x) = \prod_{i=1}^{n} (x - a_i) - 1$ is irreducible.

10: Let $p_1(x) = x^2 - 2$ and for $k \geq 2$ let $p_k(x) = p_1(p_{k-1}(x))$. Show that the roots of $p_n(x) - x$ are real and distinct for all n.

11: Let $p(x) = \sum_{i=0}^{n} a_i x^i$ with $a_0 = a_n = 1$ and $a_i > 0$ for all i. Show that if $p(x)$ has n distinct real roots then $p(2) \geq 3^n$.

12: Let $p(x)$ be the polynomial of degree n such that that $p(k) = \frac{k}{k+1}$ for $k = 0, 1, \cdots, n$. Find $p(n+1)$.

13: Find all polynomials over $\mathbf{C}$ whose coefficients are all equal to ± 1 and whose roots are all real.

14: Find all polynomials $p(x)$ over $\mathbf{R}$ such that $p(x)p(x+1) = p(x^2+x+1)$.

Putnam Problems on Polynomials

1: (1987 A4) Let $p(x,y,z)$ be a polynomial over $\mathbf{R}$ with $p(1,0,0)=4$, $p(0,1,0)=5$, $p(0,0,1)=6$, and let $f(x,y)$ be a real valued function such that $p(ux,uy,uz)=u^2f(y-x,z-x)$ for all real u,x,y,z. Given complex numbers a, b and c with $p(a,b,c)=0$ and $|b-a|=10$, find $|c-a|$.

2: (1988 A5) Show that there is a unique function $f:\mathbf{R}^+\to\mathbf{R}^+$ such that $f(f(x))=6x-f(x)$ for all $x>0$.

3: (1990 B5) Determine whether there exists a sequence $a_0,a_1,a_2\cdots$ of non-zero real numbers such that for every positive integer n, the polynomial $p(x)=\sum_{i=0}^{n}a_ix^i$ has n distinct real roots.

4: (1991 A3) Find every polynomial $p(x)$ of degree $n\geq 2$ over $\mathbf{R}$ for which there exist real numbers $r_1<r_2<\cdots<r_n$ such that $p(r_i)=0$ and $p'\left(\frac{1}{2}(r_i+r_{i+1})\right)=0$ for all i.

5: (1992 B4) Let $p(x)$ be a nonzero polynomial over $\mathbf{R}$ of degree at most 1992 with no factors in common with x^3-x, and let $\dfrac{d^{1992}}{dx^{1992}}\left(\dfrac{p(x)}{x^3-x}\right)=\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials. Find the minimum possible degree for $f(x)$.

6: (1994 B2) Determine which real numbers c have the property that there is a line in the plane which intersects the curve $y=x^4+9x^3+cx^2+9x+4$ in four distinct points.

7: (1999 A2) Let $p(x)$ be a polynomial over $\mathbf{R}$ such that $p(x)\geq 0$ for all x. Show that $p(x)$ is equal to a sum of squares of polynomials over $\mathbf{R}$.

8: (1999 A5) Show that there exists a constant $c\in\mathbf{R}$ with the property that for every polynomial $p(x)$ of degree 1999 over $\mathbf{R}$ we have $|p(0)|\leq c\displaystyle\int_{-1}^{1}|p(x)|\,dx$.

9: (1999 B2) Let $p(x)$ be a polynomial of degree n over $\mathbf{C}$ such that $p''(x)$ divides $p(x)$. Show that if $p(x)$ has at least two distinct roots then it has n distinct roots.

10: (2000 A6) Let $p(x)$ be a polynomial over $\mathbf{Z}$. Let $a_0=0$ and for $n\geq 0$ let $a_{n+1}=p(a_n)$. Show that if $a_m=0$ for some $m>0$ then either $a_1=0$ or $a_2=0$.