Solutions to the Pigeonhole Principle Problems

1: Show that at any party there are two people who have the same number of friends at the party (assume that all friendships are mutual).

Solution: Let n be the number of people at the party. Each person can have $0, 1, \cdots, n-2$ or $n-1$ friends. If all n people had a different number of friends, then one person would have 0 friends and another would have $n-1$. This is not possible since the person with $n-1$ friends is friends with everyone at the party, including the person with 0 friends.

2: Show that if 9 distinct points are chosen in the integer lattice $\mathbf{Z}^3$, then the line segment between some two of the 9 points contains another point in $\mathbf{Z}^3$.

Solution: Reduce each of the 9 points modulo 2 to get 9 points in ${\mathbf{Z}_2}^3$. Since ${\mathbf{Z}_2}^3$ only has 8 points, two of the 9 points, say a and b, must be equal as elements of ${\mathbf{Z}_2}^3$, and so $a + b = 0 \in {\mathbf{Z}_2}^3$. This means that all three entries of $a + b \in \mathbf{Z}^3$ are even and so the midpoint $\frac{1}{2}(a + b)$ lies in $\mathbf{Z}^3$.

3: Let S be a set of n integers. Show that there is a subset of S, the sum of whose elements is a multiple of n.

Solution: Let $S = \{a_1, a_2, \cdots, a_n\}$. For each $k = 1, 2, \cdots, n$, let $s_k = a_1 + a_2 + \cdots + a_k$. Reduce each of the numbers s_k modulo n to get $s_k \in \mathbf{Z}_n$. If some $s_k = 0 \in \mathbf{Z}_n$ then the sum s_k is a multiple of n in $\mathbf{Z}$. Otherwise, two of the sums s_k must be equal in $\mathbf{Z}_n$, say $s_k = s_l$ with $l > k$. Then $a_{k+1}+a_{k+2}+\cdots+a_l = s_l - s_k = 0 \in \mathbf{Z}_n$, so the sum $a_{k+1} + a_{k+2} + \cdots + a_l$ is a multiple of n in $\mathbf{Z}$.

4: Show that if 101 integers are chosen from the set $\{1, 2, 3, \cdots, 200\}$ then one of the chosen integers divides another.

Solution: Let the chosen integers be $a_1, a_2, \cdots, a_{101}$. For each $k = 1, 2, \cdots, 101$ write $a_k = 2^{m_k} b_k$ with b_k odd. The 101 numbers $b_1, b_2, \cdots, b_n$ are all odd and lie in the 100-element set $\{1, 3, 5, \cdots, 199\}$, and so some pair of the numbers b_k must be equal. Say $b_k = b_l$ with $m_k \leq m_l$. Then a_k divides a_l.

5: Show that for some integer $k > 1$, 3^k ends with 0001 (in its decimal representation).

Solution: Reduce each of the 10001 numbers $3^1, 3^2, 3^3, \cdots, 3^{10001}$ modulo 10000. Some pair of powers 3^k must be equal in $\mathbf{Z}_{10000}$, say $3^k = 3^l \in \mathbf{Z}_{10000}$ with $k < l$. Since 3 is invertible in $\mathbf{Z}_{10000}$ we have $3^{l-k} = 1 \in \mathbf{Z}_{10000}$ and so in $\mathbf{Z}$, the number 3^{l-k} ends with 0001.

6: Let n be a positive integer. Show that there is a positive multiple of n whose digits (in the base 10 representation) are all 0's and 1's.

Solution: Let $a_1 = 1$, $a_2 = 11$, $a_3 = 111$, $a_4 = 1111$ and so on. Consider the $n+1$ numbers $a_1, a_2, \cdots a_{n+1}$ all reduced modulo n. Some 2 of these must be equal in $\mathbf{Z}_n$, say $a_k = a_l \in \mathbf{Z}_n$ with $k < l$. Then $a_l - a_k = 0 \in \mathbf{Z}_n$ so $a_l - a_k$ is a multiple of n, and notice that $a_l - a_k$ is of the form $11 \cdots 100 \cdots 0$.

7: Show that some pair of any 5 points in the unit square will be at most $\frac{\sqrt{2}}{2}$ units apart, and that some pair of any 8 points in the unit square will be at most $\frac{\sqrt{5}}{4}$ units apart.

Solution: Place the square with its vertices at $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$. The unit square can be covered by 4 closed discs, each of diameter $\frac{\sqrt{2}}{2}$, with centers at $\left(\pm\frac{1}{4}, \pm\frac{1}{4}\right)$. When 5 points are placed on the square, some two of them must lie in the same disc, and these two points will be at most $\frac{\sqrt{2}}{2}$ units apart. The unit square can also be covered by 7 closed discs, each of radius $\frac{\sqrt{5}}{4}$, with centers at $\left(\pm\frac{1}{4}, \pm\frac{3}{8}\right)$. $\left(\pm\frac{1}{2}, 0\right)$ and $(0, 0)$. When 8 points are placed in the unit square, some pair of them must lie in the same disc.

8: A salesman sells at least 1 car each day for 100 consecutive days selling a total of 150 cars. Show that for each value of n with $1 \leq n < 50$, there is a period of consecutive days during which he sold a total of exactly n cars.

Solution: Let n be an integer with $1 \leq n < 50$. For $k = 1, 2, \cdots 100$, let a_k be the number of cars sold on the k^{th} day, and let $s_k = a_1 + a_2 + \cdots + a_k$. Notice that $s_1 < s_2 < s_3 < \cdots < s_{100} = 150$ and that $(s_1 + n) < (s_2 + n) < (s_3 + n) < \cdots < (s_{100} + n) = 150 + n < 200$. Two of the 200 numbers in the set $\{s_1, s_2, \cdots, s_{100}\} \cup \{(s_1 + n), (s_2 + n), \cdots, (s_{100} + n)\}$ must be equal. No two of the numbers s_k are equal and no two of the of the numbers $(s_l + n)$ are equal, and so we must have $s_k = s_l + n$ for some k, l. Then we have $k > l$ and $a_{l+1} + a_{l+2} + \cdots + a_k = s_k - s_l = n$.

9: Show that there is a Fibonacci number that ends with 9999 (in its base 10 representation).

Solution: The Fibonacci numbers are $a_0 = 0$, $a_1 = 1$, $a_2 = 1$, $a_2 = 2$, $a_3 = 3$ and so on with $a_{n+2} = a_{n+1} + a_n$. We can also use the formula $a_n = a_{n+2} - a_{n+1}$ to extend the sequence to include negative terms $a_{-1} = 1$, $a_{-2} = -1$, $a_{-3} = 2$ and so on. Reduce all the (infinitely many) pairs (a_k, a_{k+1}) modulo 10000. Some two of these pairs must be equal in ${\mathbf{Z}_{10000}}^2$, say $(a_k, a_{k+1}) = (a_l, a_{l+1}) \in {\mathbf{Z}_{10000}}^2$ with $k < l$. From the recursion formula $a_{n+2} = a_{n+1} + a_n$ we see that $a_{k+i} = a_{l+i} \in \mathbf{Z}_{10000}$ for all $i \geq 0$. From the recursion formula $a_n = a_{n+2} - a_{n+1}$ we also see that $a_{k+i} = a_{l+i} \in \mathbf{Z}_{10000}$ for all $i < 0$. Thus the Fibonacci sequence is periodic in $\mathbf{Z}_{10000}$, indeed we have $a_i = a_{l-k+i} \in \mathbf{Z}_{10000}$ for all $i \in \mathbf{Z}$. In particular, $a_{l-k-2} = a_{-2} = -1 \in \mathbf{Z}_{10000}$, and so a_{l-k-2} ends with the digits 9999.

10: Determine whether the sequence $\left\{ \dfrac{1}{n \sin n} \right\}$ converges.

Solution: For each positive integer k we can find an integer $n_k \in \left[2\pi k + \frac{\pi}{3}, 2\pi k + \frac{2\pi}{3}\right]$ (since this interval is of size $\frac{\pi}{3} > 1$). Then $n_k > 2\pi k$ and $\sin n_k \geq \frac{\sqrt{3}}{2}$, and so $n_k \sin n_k \geq \sqrt{3}\pi k$, hence $\frac{1}{n_k \sin n_k} \leq \frac{1}{\sqrt{3}\pi k} \to 0$ as $k \to \infty$. This shows that if the sequence $\left\{\frac{1}{n \sin n}\right\}$ does converge, then its limit must be zero.

For an integer k, let $\overline{k}$ denote the real number with $\overline{k} \in [0, \pi)$ such that $\overline{k} = k + \pi l$ for some integer l. Note that for any positive integer m, we can find n with $1 \leq n \leq m$ such that $\overline{n} \in \left[0, \frac{\pi}{m}\right) \cup \left[\pi - \frac{\pi}{m}, \pi\right)$; indeed if none of the m numbers $\overline{1}, \overline{2}, \overline{3}, \cdots \overline{m}$ were in the interval $\left[0 \frac{\pi}{m}\right)$, then one of the $m - 1$ intervals $\left[\frac{\pi}{m}, \frac{2\pi}{m}\right), \left[\frac{2\pi}{m}, \frac{3\pi}{m}\right), \left[\frac{3\pi}{m}, \frac{4\pi}{m}\right), \cdots \left[\frac{(n-1)\pi}{m}, \frac{\pi}{m}\right)$ would contain two of the numbers $\overline{1}, \overline{2}, \overline{3}, \cdots, \overline{m}$, and if say $\overline{n_1}$ and $\overline{n_2}$ were in the same interval with $n_1 < n_2$, then we could take $n = n_2 - n_1$ and then $\overline{n} \in \left[0, \frac{\pi}{m}\right) \cup \left[\pi - \frac{\pi}{m}, \pi\right)$.

Choose $m_1 = 1$ and $n_1 = 1$. Having chosen m_k and n_k with $1 \leq n_k \leq m_k$ and $\overline{n_k} \in \left[0, \frac{\pi}{m_k}\right) \cup \left[\pi - \frac{\pi}{m_k}, \pi\right)$, choose m_{k+1} large enough so that $\frac{\pi}{m_{k+1}} < \min\{\overline{1}, \pi - \overline{1}, \overline{2}, \pi - \overline{2}, \overline{3}, \pi - \overline{3}, \cdots, \overline{m_k}, \pi - \overline{m_k}\}$, then choose n_{k+1} with $1 \leq n_{k+1} \leq m_{k=1}$ so that $\overline{n_{k+1}} \in \left[0, \frac{\pi}{m_{k+1}}\right]$. Our choice of m_{k+1} ensures that $\overline{n_{k+1}} \notin \{\overline{1}, \overline{2}, \overline{3}, \cdots, \overline{n_k}\}$ so that $n_{k+1} > n_k$. Also, we have $n_k \leq m_k$ and $|\sin n_k| = \sin \overline{n_k} \leq \sin \frac{\pi}{m_k} \leq \frac{\pi}{m_k}$, and so $\left|\frac{1}{n_k \sin n_k}\right| \geq \frac{1}{\pi}$. This implies that the limit of the sequence $\left\{\frac{1}{n \sin n}\right\}$ cannot be 0, so it diverges.