# Lesson 1: The Pigeonhole Principle

**1:** Show that at any party there are two people who have the same number of friends at the party (assume that all friendships are mutual).

**2:** Show that if 9 distinct points are chosen in the integer lattice $\mathbf{Z}^3$, then the line segment between some two of the 9 points contains another point in $\mathbf{Z}^3$.

**3:** Let $S$ be a set of $n$ integers. Show that there is a subset of $S$, the sum of whose elements is a multiple of $n$.

**4:** Show that if 101 integers are chosen from the set $\{1, 2, 3, \cdots, 200\}$ then one of the chosen integers divides another.

**5:** Show that for some integer $k > 1$, $3^k$ ends with 0001 (in its decimal representation).

**6:** Let $n$ be a positive integer. Show that there is a positive multiple of $n$ whose digits (in the base 10 representation) are all 0's and 1's.

**7:** Show that some pair of any 5 points in the unit square will be at most $\frac{\sqrt{2}}{2}$ units apart, and that some pair of any 8 points in the unit square will be at most $\frac{\sqrt{5}}{4}$ units apart.

**8:** A salesman sells at least 1 car each day for 100 consecutive days selling a total of 150 cars. Show that for each value of $n$ with $1 \leq n < 50$, there is a period of consecutive days during which he sold a total of exactly $n$ cars.

**9:** Show that there is a Fibonacci number that ends with 9999 (in its base 10 representation).

**10:** Determine whether the sequence $\left\{ \dfrac{1}{n \sin n} \right\}$ converges.

# Putnam Problems Involving the Pigeonhole Principle

**1:** (1989 A5) Let $n \in \mathbf{Z}^+$. Let $P$ be a regular $(2n+1)$-gon inscribed in the unit circle. Show that there exists $c > 0$ such that for every point $p$ inside $P$, there exist two distinct vertices $u$ and $v$ of $P$ such that

$$\Big| |p-u| - |p-v| \Big| < \frac{1}{n} - \frac{c}{n^3}.$$

**2:** (1990 A3) Show that a convex pentagon with vertices in $\mathbf{Z}^2$ has area at least $\frac{5}{2}$.

**3:** (1993 A4) Let $n, m \in \mathbf{Z}^+$. Let $x_1, \ldots, x_m$ be positive integers each of which is less than or equal to $n$. Let $y_1, \ldots, y_n$ be positive integers each of which is less than or equal to $m$. Prove that there exists a nonempty sum of some $x_i$'s equal to the a sum of the $y_i$'s.

**4:** (1994 A4) Let $A, B \in M_{2\times 2}(\mathbf{Z})$. Suppose that $A + kB$ is invertible in $M_{2\times 2}(\mathbf{Z})$ for all $k \in \{0, 1, 2, 3, 4\}$. Show that $A + kB$ is invertible for all $k \in \mathbf{Z}$.

**5:** (1994 A6) Let $f_1, f_2, \cdots, f_{10} : \mathbf{Z} \to \mathbf{Z}$ be bijective maps. Suppose that for each integer $n$, there is some composite $f = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$, where $m \in \mathbf{Z}^+$ and each $i_j \in \{1, \cdots, 10\}$, with $f(0) = n$. Let

$$F = \left\{ {f_1}^{e_1} \circ {f_2}^{e_2} \circ \cdots \circ {f_{10}}^{e_{10}} \middle| \text{each } e_i \in \{0, 1\} \right\}$$

(where ${f_i}^1 = f_i$ and ${f_i}^0$ is the identity). Show that if $A$ is any nonempty finite set of integers, then at most 512 of the 1024 functions in $F$ map $A$ to itself.

**6:** (1995 B1) Let $S = \{1, 2, \cdots, 9\}$. For a partition $\alpha = \{A_1, \cdots, A_l\}$ of $S$ and an element $x \in S$, let $N(\alpha, x)$ be the number of elements in the set $A_i$ which contains $x$. Show that for any two partitions $\alpha$ and $\beta$ of $S$ there exist two distinct elements $x, y \in S$ such that $N(\alpha, x) = N(\alpha, y)$ and $N(\beta, x) = N(\beta, y)$.

**7:** (1997 B6) Find the least possible diameter of a dissection of the 3-4-5 triangle into four parts. (The diameter of a dissection is the largest of the diameters of the parts).

**8:** (1999 A5) Show that there exists a constant $c \in \mathbf{R}$ such that for every polynomial $f(x)$ of degree 1999, we have

$$|f(0| \leq c \int_{-1}^{1} |f(x)|\, dx.$$

**9:** (2000 B1) Let $A \in M_{n\times 3}(\mathbf{Z})$. Suppose that at least one entry in each row of $A$ is odd. Show that for some $x \in \mathbf{Z}^3$, at least $\frac{4n}{7}$ of the entries of $Ax$ are odd.

**10:** (2000 B6) Let $3 \leq n \in \mathbf{Z}$. Let $S \subseteq \{-1, 1\}^n = \big\{(\pm 1, \pm 1, \cdots, \pm 1)\big\} \subseteq \mathbf{R}^n$ with $|S| > \frac{2^{n+1}}{n}$. Show that there exists an equilateral triangle in $\mathbf{R}^n$ with vertices in $S$.