Week 6: Assorted Problems

1: Let $f(x) = x^2 + 6x + 7$. Find the minimum of $f(f(f(f(x))))$.

2: Flip a fair coin n times. What is the expected value for the product of the number of heads and the number of tails.

3: Let $r_1, \ldots, r_{2022}$ denote the roots of $f(x) = x^{2022} + 17x^{2021} + 1$. Let $P(x)$ be a polynomial of degree 2022 such that $P(r_j + r_j^{-1}) = 0$ for $j = 1, \ldots, 2022$. Find $P(1)/P(-1)$.

4: Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{k^2 - 2}{(k+2)!}$.

5: Prove that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} - \left(\frac{1}{\ln n}\right)^{(n+1)/n}$ is divergent.

6: Let z be a non-real complex number with $z^{23} = 1$. Compute $\displaystyle\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}}$.

7: Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square.

8: Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \cdots + a_1 x + 1$ be a polynomial with the a_i's are real numbers. Prove that $P(x)$ has a non-real complex root.

9: 1) Let x, y, z, w be distinct positive integers such that $x + y = z + w$. Prove that for any $\lambda > 1$, $x^\lambda + y^\lambda \neq z^\lambda + w^\lambda$.

2) Let p be a prime number and a, b, c, d be distinct positive integers such that $a^p + b^p = c^p + d^p$. Prove that $|a - c| + |b - d| \geq p$.

10: Evaluate $\displaystyle\int \frac{x^2 e^{\arctan x}}{\sqrt{1 + x^2}}\, dx$.

11: Define a sequence $\{a_n\}$ of positive integers as follows. Let a_1 be arbitrary. For $n \geq 1$, given $a_1, \ldots, a_n$, let a_{n+1} be the smallest positive integer that is coprime to $a_1 + \cdots + a_n$ and is distinct from $a_1, \ldots, a_n$. Prove that every positive integer appears in this sequence.

12: Let n be a positive integer. Prove that

$$\frac{1}{(n-1)!} \int_n^{\infty} (t-1)(t-2)\cdots(t-n+1)e^{-t}\, dt < \frac{1}{(e-1)^n}.$$