Week 5: Assorted Problems

1: Find the minimum value of y such that $y + x = (y - x)^2 + 3(y - x) + 3$ for some real number x.

2: Let a, b, c, x be reals with $(a + b)(b + c)(c + a) \neq 0$ and

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \quad \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \quad \frac{c^2}{c+a} = \frac{c^2}{c+b} + x.$$

Find x.

3: Let d be a positive integer and let A be a $d \times d$ matrix with integer entries such that $I_d + A + A^2 + \cdots + A^{2021} = 0$, where I_d denotes the $d \times d$ identity matrix. Determine the positive integers $n \leq 2021$ such that $A^n + A^{n+1} + \cdots + A^{2021}$ has determinant ± 1.

4: Find all positive integers n such that $n + 6(p^3 + 1)$ is a prime for all primes p with $2 \leq p < n$.

5: Find all polynomials $P(x)$ with integer coefficients such that $P'(P(x)) = P(P'(x))$.

6: Let r, s be positive integers and $a_1, \ldots, a_r, b_1, \ldots, b_s$ be positive real numbers such that

$$\sqrt[n]{a_1} + \sqrt[n]{a_2} + \cdots + \sqrt[n]{a_r} = \sqrt[n]{b_1} + \sqrt[n]{b_2} + \cdots + \sqrt[n]{b_s}$$

for infinitely many integer $n \geq 2$. Prove that $a_1 a_2 \cdots a_r = b_1 b_2 \cdots b_s$.

7: Prove that $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k!)^n}$ is irrational for all integers $n \geq 1$.

8: Let $f_0 : [0, 1] \to \mathbb{R}$ be a continuous function and let the sequence $(f_n)_{n \geq 1}$ be defined by

$$f_n(x) = \int_0^x f_{n-1}(t)\, dt$$

for $x \in [0, 1]$. Suppose $f_m(1) = \dfrac{1}{m!}$ for some integer $m \geq 1$. Show that f_0 has a fixed point.

9: Find the fraction a/m with the smallest denominator m such that its decimal expansion begins with 0.11235.

10: 1) Prove that the equation $e^r = 1 + r$ has a nonzero complex solution. $(e^{a+ib} = e^a(\cos b + i \sin b)))$

2) Prove that there exists a non-linear real-valued function $f(x)$ such that $f'(x) = f(x + 1) - f(x)$ for all $x \in \mathbb{R}$.

11: Find all pairs of positive nonzero integers a, b such that $ab - 1 \mid a^4 - 3a^2 + 1$.

12: Let $\mathbb{N}$ be the set of positive integers. For any subset S of $\mathbb{N}$, let $P(S)$ denote the set of all two-element subsets of S. Partition $P(\mathbb{N})$ arbitrarily into two sets P_1 and P_2. Prove that $\mathbb{N}$ contains an infinite subset S such that $P(S) \subset P_1$ or $P(S) \subset P_2$.