Week 4: Assorted Problems

1: Let a_n be the n-th positive integer k such that $\lfloor\sqrt{k}\rfloor \mid k$. Find a_{2021}.

2: Find all distinct positive integers $x_1, \ldots, x_n$ such that

$$1 + x_1 + 2x_1x_2 + \cdots + (n-1)x_1x_2\cdots x_{n-1} = x_1x_2\cdots x_n.$$

3: Prove that from a set of ten distinct two-digit numbers, it is possible to select two disjoint nonempty subsets whose members have the same sum.

4: Consider the sequence defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, $a_4 = 12$ and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n$$

for $n \geq 1$. Prove that $n \mid a_n$ for all $n \geq 1$.

5: Evaluate $2\cos^3(\pi/7) - \cos^2(\pi/7) - \cos(\pi/7)$.

6: Let $f : [0,1] \to [0,1]$ be a continuous function such that $f(f(f(x))) = x$ for all $x \in [0,1]$. Prove that $f(x) = x$ for all $x \in [0,1]$.

7: Let $P(x)$ be a polynomial with real coefficients of degree at least 2. Prove that if there is a real number a such that $P(a)P''(a) > (P'(a))^2$, then P has at least two nonreal zeros.

8: Find 40! given that

$$40! = abcdef\,283247\,897734\,345611\,269596\,115894\,272pqr\;stuvwx.$$

Note the letters denote unknown digits. The spaces are there to make the number more readable, and also serve as a hint.

9: Let $a_1 < \cdots < a_n$, $b_1 > \cdots > b_n$ and $\{a_1, \ldots, a_n, b_1, \ldots, b_n\} = \{1, 2, \ldots, 2n\}$. Show that

$$\sum_{i=1}^{n} |a_i - b_i|^2 = n^2.$$

10: Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that $\sum_{n=1}^{\infty} \left|1 - \frac{a_{n+1}}{a_n}\right|$ is divergent.

11: Let $P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ be a polynomial with complex coefficients such that $a_n \neq 0$ and there is an integer m such that

$$\left| \frac{a_m}{a_n} \right| > \binom{n}{m}.$$

Prove that P has at least one zero with absolute value less than 1.

12: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that f^2 and f^3 are differentiable on all of $\mathbb{R}$. Is the same true for f?