Week 3: Assorted Problems

1: Let d be a randomly chosen divisor of 2020. Find the expected value of $\dfrac{d^2}{d^2+2020}$.

2: Let $f(x)$ be a continuous real-valued function on the interval $[a,b]$ and let m_1, m_2 be real numbers such that $m_1 m_2 > 0$. Prove that the equation

$$f(x) = \frac{m_1}{a-x} + \frac{m_2}{b-x}$$

has at least a solution in the interval (a,b).

3: Prove that for all positive integers n, $n!+5$ is not a perfect square.

4: Solve the equation

$$\sin x \cos y + \sin y \cos z + \sin z \cos x = \frac{3}{2}.$$

5: Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11$, $P(11) = 17$, $P(17) = 23$.

6: Let $\tau = (1+\sqrt{5})/2$. Show that for any positive integer n, $\lfloor \tau^2 n \rfloor = \lfloor \tau \lfloor \tau n \rfloor + 1 \rfloor$.

7: Consider the sequence defined by $a_0 = a_1 = 3$ and $a_n = 7a_{n-1} - a_{n-2}$ for $n \geq 2$. Prove that $a_n - 2$ is a perfect square for all $n \geq 0$.

8: Let n be a positive integer. Let a be irrational. Prove that $\sqrt[n]{a+\sqrt{a^2-1}} + \sqrt[n]{a-\sqrt{a^2-1}}$ is irrational.

9: Prove that for different choices of signs $+$ and $-$ the expression $\pm 1 \pm 2 \pm 3 \pm \cdots \pm (4n+1)$ yields all odd positive integers less than or equal to $(2n+1)(4n+1)$.

10: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and differentiable function such that

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to\infty} f'(x) = \infty.$$

Prove that $g(x) = \sin f(x)$ is not periodic.

11: Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer and let $x^{(0)} = 1$. Prove that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$

12: Let $F(x) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)\cdots$, where the exponents are Fibonacci numbers. Show that every coefficient of $F(x)$, when expanded as a power series in x, is 0, 1 or -1.