Week 2: Assorted Problems

1: Prove that for any positive integer n,

$$n^2 + (n^2+1) + \cdots + (n^2+n) = (n^2+n+1) + (n^2+n+2) + \cdots + (n^2+2n).$$

2: For how many positive integers $n \leq 1000$ does the equation $x^{\lfloor x \rfloor} = n$ have a positive real solution?

3: Let $a = 256 = 2^8$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

4: Let n be a positive integer. Let $P(x)$ be the unique polynomial of degree n such that $P(k^2) = k$ for all $k = 0, 1, \ldots, n$. Find $P((n+1)^2)$.

5: Let $f(n)$ be a nonconstant polynomial with integer coefficients. Show that $f(n)$ is composite for infinitely many values of n.

6: Find all continuous functions $f : \mathbb{R} \to [0, \infty)$ such that

$$f(x+y)^2 - f(x-y)^2 = 4f(x)f(y)$$

for all real numbers x, y.

7: Let $f : [a, b] \to \mathbb{R}$ be an increasing continuous function. Prove that

$$(x-a)\int_x^b f(t)\,dt + (x-b)\int_a^x f(t)\,dt \geq 0$$

for all $x \in [a, b]$.

8: Find $\displaystyle\lim_{n\to\infty} \frac{2\ln 2 + 3\ln 3 + \cdots n \ln n}{n^2 \ln n}$.

9: Prove that for any integer $n \geq 0$, $f_n(x) = 1 + 2^{-n}x + 3^{-n}x^2 + 4^{-n}x^3 + 5^{-n}x^4$ has no real zero.

10: Compute the number of pairs (a, b) with $a, b \in \{0, 1, 2, 3, 4\}$ such that the maximum of $x^a(1-x)^b + (1-x)^a x^b$ is 2^{1-a-b}.

11: A sequence of positive integers $a_1, a_2, a_3, \ldots$ satisfies

$$a_{n+1} = n\left\lfloor \frac{a_n}{n} \right\rfloor + 1$$

for all positive integers n. If $a_{30} = 30$, how many possible values can a_1 take?

12: Let M be the $2^n \times n$ matrix whose rows consist of all 2^n distinct vectors of ± 1's of length n. Change any subset of the entries to 0. Show that some nonempty subset of the rows of the resulting matrix sums to the zero vector.