# Week 1: Assorted Problems

**1:** Consider the sequence $a_n$ defined by $a_0 = 1$ and $a_{n+1} = 1 + \sum_{i=1}^{n} i a_i$ for $n \geq 0$. Give a closed formula for $a_n$.

**2:** Slips of paper with the numbers from 1 to 10 are placed in a hat. Four numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order?

**3:** How many six-digit multiples of 27 have only 3, 6 or 9 as their digits?

**4:** For any positive integer $n$, let $a_n$ be the unique positive real root of $x^n + x = 1$. Find a simple function $f(x)$ such that $\lim_{n\to\infty} \frac{1 - a_n}{f(n)} = 1$.

**5:** In a triangle $ABC$ with $AB = 32$, $AC = 35$ and $BC = x$. What is the smallest positive integer $x$ such that $1 + \cos^2 A$, $\cos^2 B$ and $\cos^2 C$ form the sides of a (non-degenerate) triangle. (Here $A$ is the angle opposite to the side $BC$.)

**6:** Let $N$ be a positive integer such that there are exactly 205 ordered pairs $(x, y)$ of positive integers satisfying $\frac{1}{x} + \frac{1}{y} = \frac{1}{N}$. Prove that $N$ is a perfect square.

**7:** For any positive integer $n$, let $D(n)$ denote the last non-zero digit of $n$. Let $N$ be a positive integer and let $q$ and $r$ be the quotient and the remainder when $N$ is divided by 5. Prove that

$$D(N!) \equiv 2^q D(q!)\, r! \pmod{10}.$$

**8:** A functino $f : S \to S$ is called idempotent if $f(f(x)) = f(x)$ for all $x \in S$. Let $I_n$ be the number of idempotent functions from $\{1, 2, \ldots, n\}$ to itself. Compute $\sum_{n=1}^{\infty} \frac{I_n}{n!}$.

**9:** Let $f$ and $g$ be non-constant differentiable real-value functions on $(-\infty, \infty)$ with $f'(0) = 0$. Suppose for all $x, y \in \mathbb{R}$,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y) \\ g(x+y) &= f(x)g(y) + f(y)g(x). \end{aligned}$$

Prove that $f(x)^2 + g(x)^2 = 1$ for all $x$.

**10:** Show that the positive integers that can't be represented as a sum of two or more consecutive integers are exactly the powers of 2.

**11:** Prove that any integer strictly between 1 and $n!$ can be written as a sum of at most $n$ distinct divisors of $n!$.

**12:** Find all positive real numbers $a$ with the property that for any continuous function $f(x)$ on $[0, 1]$ with $f(0) = f(1) = 0$, there exists some $x$ with $f(x) = f(x + a)$.